Two fundamental conjectures on the stucture of Hecke algebras

Part I: The BMR

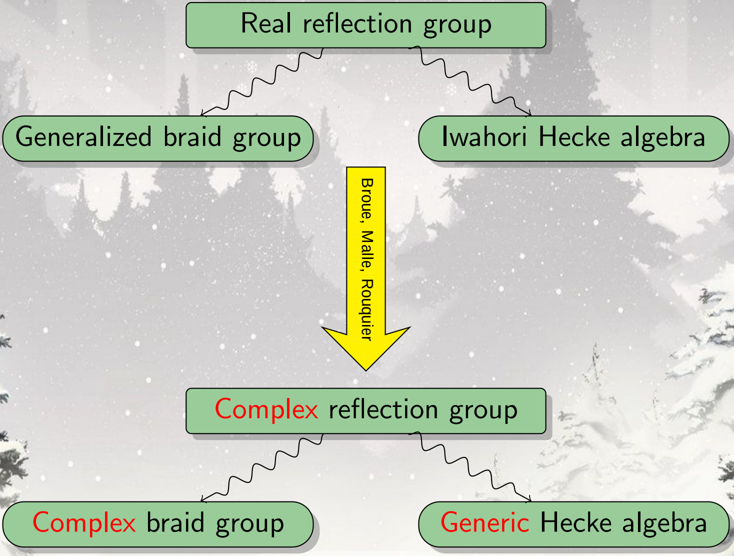
Eirini Chavli

Universität Stuttgart

8 December 2018

From real to complex

From real to complex



Complex reflection groups

A complex reflection group is a finite subgroup of $GL_n(\mathbb{C})$, which is generated by pseudo-reflections

Complex re ection groups

A complex re ection group is a nite subgroup of $GL_n(\mathbb{C})$, which is generated by pseudo-re ections i.e. elements of nite order whose vector space of xed points is a hyperplane.

Complex re ection groups

A complex re ection group is a nite subgroup of $GL_n(\mathbb{C})$, which is generated by pseudo-re ections i.e. elements of nite order whose vector space of xed points is a hyperplane.

Examples:

Complex reflection groups

A complex reflection group is a finite subgroup of $GL_n(\mathbb{C})$, which is generated by pseudo-reflections i.e. elements of finite order whose vector space of fixed points is a hyperplane.

Examples:

- Any finite Coxeter group (known also as real reflection group)

$$C = \langle s_1, \dots, s_n \mid s_i^2 = 1,\; \underbrace{s_i s_j s_i \cdots}_{m_{ij}\ \text{factors}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}\ \text{factors}} \rangle.$$

Complex re ection groups

A complex re ection group is a nite subgroup of $GL_n(\mathbb{C})$, which is generated by pseudo-re ections i.e. elements of nite order whose vector space of xed points is a hyperplane.

Examples:

Any nite Coxeter group (known also as real re ection group)

$$C = \langle s_1, \ldots, s_n \mid s_i^2 = 1;\ \underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ factors}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{ factors}} \rangle.$$

$$G_5 = \langle s_1, s_2 \mid s_1^3 = s_2^3 = 1;\quad s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 \rangle.$$

$$G_{11} = \langle s_1, s_2, s_3 \mid s_1^2 = s_2^3 = s_3^4 = 1;\quad s_1 s_2 s_3 = s_2 s_3 s_1 = s_3 s_1 s_2 \rangle.$$

Complex reflection groups

A complex reflection group is a finite subgroup of $GL_n(\mathbb{C})$, which is generated by pseudo-reflections i.e. elements of finite order whose vector space of fixed points is a hyperplane.

Examples:

Any finite Coxeter group (known also as real reflection group)

$$C = \langle s_1, \dots, s_n \mid s_i^2 = 1,\ \underbrace{s_i s_j s_i \dots}_{m_{ij} \text{ factors}} = \underbrace{s_j s_i s_j \dots}_{m_{ij} \text{ factors}} \rangle.$$

$$G_5 = \langle s_1, s_2 \mid s_1^3 = s_2^3 = 1,\ s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 \rangle.$$

$$G_{11} = \langle s_1, s_2, s_3 \mid s_1^2 = s_2^3 = s_3^4 = 1,\ s_1 s_2 s_3 = s_2 s_3 s_1 = s_3 s_1 s_2 \rangle.$$

Coxeter-like presentation:

$$\langle s_1, \dots, s_n \mid s_i^{o(s_i)} = 1,\ \text{homogenious positive relations} \rangle$$

Complex braid groups

Let $W = \left\langle s_1, \dots, s_n \;\middle|\; s_i^{o(s_i)} = 1, \text{ homogenious positive relations} \right\rangle$ be a complex reflection group.

Complex braid groups

Let $W = \langle s_1, \ldots, s_n \mid s_i^{o(s_i)} = 1,$ homogenious positive relations$\rangle$ be a complex reflection group. The group

$$B_W := \langle \sigma_1, \ldots, \sigma_n \mid \text{the same homogenious positive relations} \rangle$$

is called the complex braid group associated with W.

Complex braid groups

Let $W = \langle s_1, \ldots, s_n \mid s_i^{o(s_i)} = 1,$ homogenious positive relations$\rangle$ be a complex reflection group. The group

$$B_W := \langle \sigma_1, \ldots, \sigma_n \mid \text{the same homogenious positive relations} \rangle$$

is called the complex braid group associated with W.

Examples:

$$C = \langle s_1, \ldots, s_n \mid s_i^2 = 1, \; s_i s_j s_i \cdots = s_j s_i s_j \cdots \rangle$$
$$B_C = \langle \sigma_1, \ldots, \sigma_n \mid \sigma_i \sigma_j \sigma_i \cdots = \sigma_j \sigma_i \sigma_j \cdots \rangle$$

$$G_7 = \langle s_1, s_2, s_3 \mid s_1^2 = s_2^3 = s_3^3 = 1, \; s_1 s_2 s_3 = s_2 s_3 s_1 = s_3 s_1 s_2 \rangle$$
$$B_{G_7} = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1 \sigma_2 \sigma_3 = \sigma_2 \sigma_3 \sigma_1 = \sigma_3 \sigma_1 \sigma_2 \rangle$$

$$G_{11} = \langle s_1, s_2, s_3 \mid s_1^2 = s_2^3 = s_3^4 = 1, \; s_1 s_2 s_3 = s_2 s_3 s_1 = s_3 s_1 s_2 \rangle$$
$$B_{G_{11}} = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1 \sigma_2 \sigma_3 = \sigma_2 \sigma_3 \sigma_1 = \sigma_3 \sigma_1 \sigma_2 \rangle$$

Generic Hecke algebras

Let $(u_{s;j})_{1 \le j \le o(s)}$ be a set of indeterminates, where

Generic Hecke algebras

Let $(u_{s,j})_{1 \le j \le o(s)}$ be a set of indeterminates, where

s runs along the pseudo-reflections of W (if $s \sim s'$ then $u_{s,j} = u_{s',j}$).

Generic Hecke algebras

Let $(u_{s,j})_{1 \le j \le o(s)}$ be a set of indeterminates, where

- s runs along the pseudo-reflections of W (if $s \sim s'$ then $u_{s,j} = u_{s',j}$).
- $o(s)$ denotes the order of s.

Generic Hecke algebras

Let $(u_{s,j})_{1 \le j \le o(s)}$ be a set of indeterminates, where

- s runs along the pseudo-reflections of W (if $s \sim s'$ then $u_{s,j} = u_{s',j}$).
- $o(s)$ denotes the order of s.

Let $R_W := \mathbb{Z}[u_{s,1}^{\pm 1}, \ldots, u_{s,o(s)}^{\pm 1}]$.

Generic Hecke algebras

Let $(u_{s;j})_{1 \quad j \quad o(s)}$ be a set of indeterminates, where

- s runs along the pseudo-re ections of W (if $s \quad s^0$ then $u_{s;j} = u_{s^0;j}$).
- $o(s)$ denotes the order of s.

Let $R_W := \mathbb{Z}[u_{s;1}^{\ 1}; :::; u_{s;o(s)}^{\ 1}]$.

The generic Hecke algebra $H(W)$ associated with W is the algebra

$$R_W B_W = (\quad u_{s;1}) ::: (\quad u_{s;o(s)}):$$

Generic Hecke algebras

Let $(u_{s;j})_{1 \quad j \quad o(s)}$ be a set of indeterminates, where

- s runs along the pseudo-re ections of W (if $s \quad s'$ then $u_{s;j} = u_{s';j}$).
- $o(s)$ denotes the order of s.

Let $R_W := \mathbb{Z}[u_{s;1}^{1}; \ldots; u_{s;o(s)}^{1}]$.

The generic Hecke algebra $H(W)$ associated with W is the algebra

$$R_W B_W = (\quad u_{s;1}) \cdots (\quad u_{s;o(s)}) :$$

Examples:

Generic Hecke algebras

Let $(u_{s,j})_{1 \le j \le o(s)}$ be a set of indeterminates, where

- s runs along the pseudo-reflections of W (if $s \sim s'$ then $u_{s,j} = u_{s',j}$).
- $o(s)$ denotes the order of s.

Let $R_W := \mathbb{Z}[u_{s,1}^{\pm 1}, \ldots, u_{s,o(s)}^{\pm 1}]$.

The generic Hecke algebra $H(W)$ associated with W is the algebra

$$R_W B_W / ((\sigma - u_{s,1}) \cdots (\sigma - u_{s,o(s)})).$$

Examples:

$G_4 = \langle s_1, s_2 \mid s_1^3 = s_2^3 = 1, s_1 s_2 s_1 = s_2 s_1 s_2 \rangle$

Generic Hecke algebras

Let $(u_{s;j})_{1 \quad j \quad o(s)}$ be a set of indeterminates, where

- s runs along the pseudo-reflections of W (if $s \quad s^0$ then $u_{s;j} = u_{s^0;j}$).
- $o(s)$ denotes the order of s.

Let $R_W := Z[u_{s;1}^{1}; :::; u_{s;o(s)}^{1}]$.

The generic Hecke algebra $H(W)$ associated with W is the algebra

$$R_W B_W = (\quad u_{s;1}) ::: (\quad u_{s;o(s)}):$$

Examples:

$G_4 \quad = \quad h s_1; s_2 \; j \; s_1^3 = s_2^3 = 1; s_1 s_2 s_1 = s_2 s_1 s_2 i$

$H_{G_4} \quad = \quad h \quad _1; \quad _2 \; j \quad _1 \quad _2 \quad _1 = \quad _2 \quad _1 \quad _2;$

Generic Hecke algebras

Let $(u_{s,j})_{1 \le j \le o(s)}$ be a set of indeterminates, where

- s runs along the pseudo-reflections of W (if $s \sim s'$ then $u_{s,j} = u_{s',j}$).
- $o(s)$ denotes the order of s.

Let $R_W := \mathbb{Z}[u_{s,1}^{\pm 1}, \ldots, u_{s,o(s)}^{\pm 1}]$.

The generic Hecke algebra $H(W)$ associated with W is the algebra

$$R_W B_W / ((\sigma - u_{s,1}) \cdots (\sigma - u_{s,o(s)})).$$

Examples:

$$G_4 = \langle s_1, s_2 \mid s_1^3 = s_2^3 = 1, s_1 s_2 s_1 = s_2 s_1 s_2 \rangle$$

$$H_{G_4} = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2, \prod_{j=1}^{3} (\sigma_1 - u_j) = \prod_{j=1}^{3} (\sigma_2 - u_j) = 0 \rangle$$

Generic Hecke algebras

Let $(u_{s;j})_{1 \; j \; o(s)}$ be a set of indeterminates, where

- s runs along the pseudo-re ections of W (if $s \quad s^0$ then $u_{s;j} = u_{s^0;j}$).
- $o(s)$ denotes the order of s.

Let $R_W := \mathbb{Z}[u_{s;1}^{\;1}; \dots; u_{s;o(s)}^{\;1}]$.

The generic Hecke algebra $H(W)$ associated with W is the algebra

$$R_W B_W = (\quad u_{s;1}) \cdots (\quad u_{s;o(s)}):$$

Examples:

$$G_4 = \langle s_1; s_2 \mid s_1^3 = s_2^3 = 1; s_1 s_2 s_1 = s_2 s_1 s_2 \rangle$$

$$H_{G_4} = \langle \;_1; \;_2 \mid \;_1 \;_2 \;_1 = \;_2 \;_1 \;_2; \prod_{j=1}^{3} (\;_1 \quad u_j) = \prod_{j=1}^{3} (\;_2 \quad u_j) = 0 \rangle$$

$$G_6 = \langle s_1; s_2 \mid s_1^2 = s_2^3 = 1; (s_1 s_2)^3 = (s_2 s_1)^3 \rangle$$

$$H_{G_6} = \langle \;_1; \;_2 \mid (\;_1 \;_2)^3 = (\;_2 \;_1)^3; \prod_{j=1}^{2} (\;_1 \quad u_{s_1;j}) = \prod_{j=1}^{3} (\;_2 \quad u_{s_2;j}) = 0 \rangle$$

The BMR freeness conjecture

Conjecture (Broué-Malle-Rouquier 1998)

$H(W)$ is a free R_W-module of rank $|W|$.

The BMR freeness conjecture

Conjecture (Broué-Malle-Rouquier 1998)

$H(W)$ is a free R_W-module of rank $|W|$.

Proposition (Broué-Malle-Rouquier 1998)

If $H(W)$ is spanned over R_W by $|W|$ elements, then it is a free R_W-module of rank $|W|$.

The BMR freeness conjecture

Conjecture (Broué-Malle-Rouquier 1998)

$H(W)$ is a free R_W-module of rank $|W|$.

Proposition (Broué-Malle-Rouquier 1998)

If $H(W)$ is spanned over R_W by $|W|$ elements, then it is a free R_W-module of rank $|W|$.

However...

Every complex reflection group is a direct product of irreducible ones.

The BMR freeness conjecture

Conjecture (Broué-Malle-Rouquier 1998)

$H(W)$ is a free R_W-module of rank $|W|$.

Proposition (Broué-Malle-Rouquier 1998)

If $H(W)$ is spanned over R_W by $|W|$ elements, then it is a free R_W-module of rank $|W|$.

However...

Every complex reflection group is a direct product of irreducible ones.

Conjecture (equivalent form)

$H(W)$ is spanned over R_W by $|W|$ elements, where W is an irreducible complex reflection group.

The Shephard Todd classi cation

The nite irreducible complex re ection groups were classi ed by Shephard and Todd (1954). They belong either to

The Shephard Todd classi cation

The nite irreducible complex re ection groups were classi ed by Shephard and Todd (1954). They belong either to

- The in nite family $G(de; e; n)$ of groups of n n monomial matrices

The Shephard Todd classification

The finite irreducible complex reflection groups were classified by Shephard and Todd (1954). They belong either to

- The infinite family $G(de, e, n)$ of groups of $n \times n$ monomial matrices such that their non-zero coefficients belong to $\mu_{de}(\mathbb{C})$

The Shephard Todd classification

The finite irreducible complex reflection groups were classified by Shephard and Todd (1954). They belong either to

- The infinite family $G(de, e, n)$ of groups of $n \times n$ monomial matrices such that their non-zero coefficients belong to $\mu_{de}(\mathbb{C})$ and their product to $\mu_d(\mathbb{C})$.

The Shephard Todd classification

The finite irreducible complex reflection groups were classified by Shephard and Todd (1954). They belong either to

- The infinite family $G(de, e, n)$ of groups of $n \times n$ monomial matrices such that their non-zero coefficients belong to $\mu_{de}(\mathbb{C})$ and their product to $\mu_d(\mathbb{C})$.
- The finite set of 34 exceptions $G_4, G_5, \ldots, G_{37}$.

The Shephard–Todd classification

The finite irreducible complex reflection groups were classified by Shephard and Todd (1954). They belong either to

- The infinite family $G(de, e, n)$ of groups of $n \times n$ monomial matrices such that their non-zero coefficients belong to $\mu_{de}(\mathbb{C})$ and their product to $\mu_d(\mathbb{C})$.
- The finite set of 34 exceptions $G_4, G_5, \ldots, G_{37}$.

Among the irreducible complex reflection groups we encounter the irreducible finite Coxeter groups.

The Shephard Todd classification

The finite irreducible complex reflection groups were classified by Shephard and Todd (1954). They belong either to

- The infinite family $G(de, e, n)$ of groups of $n \times n$ monomial matrices such that their non-zero coefficients belong to $\mu_{de}(\mathbb{C})$ and their product to $\mu_d(\mathbb{C})$.
- The finite set of 34 exceptions $G_4, G_5, \ldots, G_{37}$.

Among the irreducible complex reflection groups we encounter the irreducible finite Coxeter groups. More precisely:

$G(1,1,n) = A_{n-1}$, $G(2,1,n) = B_n$, $G(2,2,n) = D_n$, $G(m,m,2) = I_2(m)$, $G_{23} = H_3$, $G_{28} = F_4$, $G_{30} = H_4$, $G_{35} = E_6$, $G_{36} = E_7$, $G_{37} = E_8$.

The BMR freeness conjecture

When stated, this conjecture was already known to hold for real reflection groups (Bourbaki), for $G(de, e, n)$ (Ariki, Ariki-Koike[a]) and for G_4 (Broué-Malle, Berceanu - Funar).

[a] an altenative basis for $G(e, e, n)$, $G(d, 1, n)$ was provided by Neaime (2018)

The BMR freeness conjecture

When stated, this conjecture was already known to hold for real reflection groups (Bourbaki), for $G(de, e, n)$ (Ariki, Ariki-Koike[a]) and for G_4 (Broué-Malle, Berceanu - Funar).

[a] an altenative basis for $G(e, e, n)$, $G(d, 1, n)$ was provided by Neaime (2018)

Theorem: The BMR freeness conjecture is true.

The BMR freeness conjecture

When stated, this conjecture was already known to hold for real re ection group (Bourbaki), for $G(de;e;n)$ (Ariki, Ariki-Koike[a]) and for G_4 (Broue-Malle, Berceanu - Funar).

[a] an altenative basis for $G(e;e;n)$, $G(d;1;n)$ was provided by Neaime (2018)

Theorem: The BMR freeness conjecture is true.

- G_4, G_{25}, G_{32} (Marin, 2011)
- G_{26} (Marin, 2012)
- $G_{12}; G_{22}; G_{24}; G_{27}; G_{29}; G_{31}; G_{33}; G_{34}$ (Marin-Pfei er 2014)
- $G_8; G_{16}$ (C. 2015)
- $G_4; \dots; G_{15}$ (C. 2016)
- $G_{20}; G_{21}$ (Marin 2017)
- $G_{17}; G_{18}; G_{19}$ (Tsuchioka 2017)

The BMR freeness conjecture

When stated, this conjecture was already known to hold for real re ection group (Bourbaki), for $G(de; e; n)$ (Ariki, Ariki-Koike[a]) and for G_4 (Broue-Malle, Berceanu - Funar).

[a] an altenative basis for $G(e; e; n)$, $G(d; 1; n)$ was provided by Neaime (2018)

Theorem: The BMR freeness conjecture is true.

- G_4, G_{25}, G_{32} (Marin, 2011)
- G_{26} (Marin, 2012)
- $G_{12}; G_{22}; G_{24}; G_{27}; G_{29}; G_{31}; G_{33}; G_{34}$ (Marin-Pfei er 2014)
- $G_8; G_{16}$ (C. 2015)
- $G_4; \ldots; G_{15}$ (C. 2016)
- $G_{20}; G_{21}$ (Marin 2017)
- $G_{17}; G_{18}; G_{19}$ (Tsuchioka 2017)

The exceptional groups of rank 2

The groups $G_4, \ldots, G_{22}$, known as the exceptional groups of rank 2, fall into three families:

The exceptional groups of rank 2

The groups $G_4; \ldots; G_{22}$, known as the exceptional groups of rank 2, fall into three families:

- Tetrahedral family $G_4; \ldots; G_7$:

The exceptional groups of rank 2

The groups $G_4; \ldots; G_{22}$, known as the exceptional groups of rank 2, fall into three families:

Tetrahedral family $G_4; \ldots; G_7$: For $n = 4; \ldots; 7$ we have $G_n / Z(G_n) \simeq A_4$.

The exceptional groups of rank 2

The groups $G_4, \ldots, G_{22}$, known as the exceptional groups of rank 2, fall into three families:

- Tetrahedral family $G_4, \ldots, G_7$: For $n = 4, \ldots, 7$ we have $G_n/Z(G_n) \simeq A_4$.
- Octahedral family $G_8, \ldots, G_{15}$: For $n = 8, \ldots, 15$ we have $G_n/Z(G_n) \simeq S_4$.
- Icosahedral family $G_{16}, \ldots, G_{22}$: For $n = 16, \ldots, 22$ we have $G_n/Z(G_n) \simeq A_5$.

The exceptional groups of rank 2

The groups $G_4, \ldots, G_{22}$, known as the exceptional groups of rank 2, fall into three families:

Tetrahedral family $G_4, \ldots, G_7$: For $n = 4, \ldots, 7$ we have $G_n/Z(G_n) \simeq A_4$.

Octahedral family $G_8, \ldots, G_{15}$: For $n = 8, \ldots, 15$ we have $G_n/Z(G_n) \simeq S_4$.

Icosahedral family $G_{16}, \ldots, G_{22}$: For $n = 16, \ldots, 22$ we have $G_n/Z(G_n) \simeq A_5$.

We know that $G_n/Z(G_n)$ is the group of even elements in a finite Coxeter group C of rank 3 of type A_3, B_3, H_3, respectively, with Coxeter matrix (m_{ij}).

The exceptional groups of rank 2

The groups $G_4, \dots, G_{22}$, known as the exceptional groups of rank 2, fall into three families:

- Tetrahedral family $G_4, \dots, G_7$: For $n = 4, \dots, 7$ we have $G_n/Z(G_n) \cong A_4$.
- Octahedral family $G_8, \dots, G_{15}$: For $n = 8, \dots, 15$ we have $G_n/Z(G_n) \cong S_4$.
- Icosahedral family $G_{16}, \dots, G_{22}$: For $n = 16, \dots, 22$ we have $G_n/Z(G_n) \cong A_5$.

We know that $G_n/Z(G_n)$ is the group of even elements in a finite Coxeter group C of rank 3 of type A_3, B_3, H_3, respectively, with Coxeter matrix (m_{ij}).

Let $t_{ij,k}$ be set of indeterminates, where $i \neq j$, $k \in \mathbb{Z}/m_{ij}\mathbb{Z}$ and $t_{ij,k}^{-1} = t_{ji,-k}$. Let $\tilde{R} := \mathbb{Z}[t_{ij,k}]$.

The exceptional groups of rank 2

The groups $G_4, \ldots, G_{22}$, known as the exceptional groups of rank 2, fall into three families:

Tetrahedral family $G_4, \ldots, G_7$: For $n = 4, \ldots, 7$ we have $G_n/Z(G_n) \simeq A_4$.

Octahedral family $G_8, \ldots, G_{15}$: For $n = 8, \ldots, 15$ we have $G_n/Z(G_n) \simeq S_4$.

Icosahedral family $G_{16}, \ldots, G_{22}$: For $n = 16, \ldots, 22$ we have $G_n/Z(G_n) \simeq A_5$.

We know that $G_n/Z(G_n)$ is the group of even elements in a finite Coxeter group C of rank 3 of type A_3, B_3, H_3, respectively, with Coxeter matrix (m_{ij}).

Let $t_{ij,k}$ be set of indeterminates, where $i \neq j$, $k \in \mathbb{Z}/m_{ij}\mathbb{Z}$ and $t_{ij,k}^{-1} = t_{ji,-k}$. Let $\tilde{R} := \mathbb{Z}[t_{ij,k}]$.

$A(C) = \langle Y_1, Y_2, Y_3 \mid Y_i^2 = 1,$

The exceptional groups of rank 2

The groups $G_4, \ldots, G_{22}$, known as the exceptional groups of rank 2, fall into three families:

- Tetrahedral family $G_4, \ldots, G_7$: For $n = 4, \ldots, 7$ we have $G_n/Z(G_n) \simeq A_4$.
- Octahedral family $G_8, \ldots, G_{15}$: For $n = 8, \ldots, 15$ we have $G_n/Z(G_n) \simeq S_4$.
- Icosahedral family $G_{16}, \ldots, G_{22}$: For $n = 16, \ldots, 22$ we have $G_n/Z(G_n) \simeq A_5$.

We know that $G_n/Z(G_n)$ is the group of even elements in a finite Coxeter group C of rank 3 of type A_3, B_3, H_3, respectively, with Coxeter matrix (m_{ij}).

Let $t_{ij;k}$ be set of indeterminates, where $i \neq j$, $k \in \mathbb{Z}/m_{ij}\mathbb{Z}$ and $t_{ij;k}^{-1} = t_{ji;-k}$. Let $\tilde{R} := \mathbb{Z}[t_{ij;k}]$.

$$A(C) = \langle Y_1, Y_2, Y_3 \mid Y_i^2 = 1,\ i \neq j \text{ implies } \prod_{k=1}^{m_{ij}} (Y_i Y_j - t_{ij;k}) = 0 \rangle$$

The exceptional groups of rank 2

The groups $G_4, \ldots, G_{22}$, known as the exceptional groups of rank 2, fall into three families:

- Tetrahedral family $G_4, \ldots, G_7$: For $n = 4, \ldots, 7$ we have $G_n/Z(G_n) \simeq A_4$.
- Octahedral family $G_8, \ldots, G_{15}$: For $n = 8, \ldots, 15$ we have $G_n/Z(G_n) \simeq S_4$.
- Icosahedral family $G_{16}, \ldots, G_{22}$: For $n = 16, \ldots, 22$ we have $G_n/Z(G_n) \simeq A_5$.

We know that $G_n/Z(G_n)$ is the group of even elements in a finite Coxeter group C of rank 3 of type A_3, B_3, H_3, respectively, with Coxeter matrix (m_{ij}).

Let $t_{ij;k}$ be set of indeterminates, where $i \neq j$, $k \in \mathbb{Z}/m_{ij}\mathbb{Z}$ and $t_{ij;k}^{-1} = t_{ji;-k}$. Let $\tilde{R} := \mathbb{Z}[t_{ij;k}]$.

$$A(C) = \langle Y_1, Y_2, Y_3 \mid Y_i^2 = 1,\ i \neq j \text{ implies } \prod_{k=1}^{m_{ij}} (Y_i Y_j - t_{ij;k}) = 0 \rangle$$

$$A_+(C) = \langle A_{ij} := Y_i Y_j,\ i \neq j \rangle$$

Etingof-Rains 2004

For every reduced word w_x that represents x inside $G_n = Z(G_n)$, we denote by T_{w_x} the corresponding element in $A_+(C)$.

Etingof-Rains 2004

For every reduced word w_x that represents x inside $G_n = Z(G_n)$, we denote by T_{w_x} the corresponding element in $A_+(C)$.

Example: Let C be the finite Coxeter group of type B_3. Let $w_x = y_1 y_2 y_1 y_3$.

Etingof-Rains 2004

For every reduced word w_x that represents x inside $G_n = Z(G_n)$, we denote by T_{w_x} the corresponding element in $A_+(C)$.

Example: Let C be the finite Coxeter group of type B_3. Let $w_x = y_1y_2y_1y_3$. Then, $T_{w_x} = A_{12}A_{13}$.

Etingof-Rains 2004

For every reduced word w_x that represents x inside $G_n/Z(G_n)$, we denote by T_{w_x} the corresponding element in $A_+(C)$.

Example: Let C be the finite Coxeter group of type B_3. Let $w_x = y_1 y_2 y_1 y_3$. Then, $T_{w_x} = A_{12} A_{13}$.

Proposition (Etingof-Rains 2004)

$A_+(C)$ is generated as $\mathcal{R}$-module by the elements T_{w_x}, $x \in G_n/Z(G_n)$.

Etingof-Rains 2004

For every reduced word w_x that represents x inside $G_n/Z(G_n)$, we denote by T_{w_x} the corresponding element in $A_+(C)$.

Example: Let C be the finite Coxeter group of type B_3. Let $w_x = y_1 y_2 y_1 y_3$. Then, $T_{w_x} = A_{12} A_{13}$.

Proposition (Etingof-Rains 2004)

$A_+(C)$ is generated as $\tilde{R}$-module by the elements T_{w_x}, $x \in G_n/Z(G_n)$.

Let $R^+ = R_{G_n}[z, z^{-1}]$, where z is the image of the generator of the center of G_n inside $H(G_n)$.

Etingof-Rains 2004

For every reduced word w_x that represents x inside $G_n/Z(G_n)$, we denote by T_{w_x} the corresponding element in $A_+(C)$.

Example: Let C be the finite Coxeter group of type B_3. Let $w_x = y_1 y_2 y_1 y_3$. Then, $T_{w_x} = A_{12} A_{13}$.

Proposition (Etingof-Rains 2004)

$A_+(C)$ is generated as $\tilde{R}$-module by the elements T_{w_x}, $x \in G_n/Z(G_n)$.

Let $R^+ = R_{G_n}[z, z^{-1}]$, where z is the image of the generator of the center of G_n inside $H(G_n)$.

Proposition (Etingof-Rains 2004)

There is a ring morphism $: \tilde{R} \to R^+$, inducing $: A_+(C) \quad R^+ \quad H(G_n)$, considering $H(G_n)$ as R^+-module.

Etingof-Rains 2004

For every reduced word w_x that represents x inside $G_n = Z(G_n)$, we denote by T_{w_x} the corresponding element in $A_+(C)$.

Example: Let C be the nite Coxeter group of type B_3. Let $w_x = y_1 y_2 y_1 y_3$. Then, $T_{w_x} = A_{12} A_{13}$.

Proposition (Etingof-Rains 2004)

$A_+(C)$ is generated as R-module by the elements T_{w_x}, x 2 $G_n = Z(G_n)$.

Let $R^+ = R_{G_n}[z; z^{1}]$, where z is the image of the generator of the center of G_n inside $H(G_n)$.

Proposition (Etingof-Rains 2004)

There is a ring morphism : R R^+, inducing : $A_+(C)$ R^+ $H(G_n)$, considering $H(G_n)$ as R^+-module.

(A_{ij}) = product of $_i$'s

The proof of BMR for the first two families

1. For every $x \in G_n/Z(G_n)$, $n = 4, \ldots, 15$, we fix a reduced word w_x in letters y_1, y_2 and y_3 that represents x in $G_n/Z(G_n)$.

The proof of BMR for the first two families

1. For every $x \in G_n/Z(G_n)$, $n = 4, \ldots, 15$, we fix a reduced word w_x in letters y_1, y_2 and y_3 that represents x in $G_n/Z(G_n)$.
   $w_x = y_2 y_3$ for $C = B_3$

The proof of BMR for the first two families

1. For every $x \in G_n/Z(G_n)$, $n = 4, \ldots, 15$, we fix a reduced word w_x in letters y_1, y_2 and y_3 that represents x in $G_n/Z(G_n)$.
   $w_x = y_2 y_3$ for $C = B_3$
2. From the reduced word w_x one can obtain a word $\tilde{w_x}$ not necessarily reduced, that also represents x in $G_n/Z(G_n)$, such that

The proof of BMR for the first two families

1. For every $x \in G_n/Z(G_n)$, $n = 4, \ldots, 15$, we fix a reduced word w_x in letters y_1, y_2 and y_3 that represents x in $G_n/Z(G_n)$.
   $w_x = y_2 y_3$ for $C = B_3$
2. From the reduced word w_x one can obtain a word $\tilde{w}_x$ not necessarily reduced, that also represents x in $G_n/Z(G_n)$, such that

   $$\tilde{w}_1 = 1.$$

The proof of BMR for the first two families

1. For every $x \in G_n/Z(G_n)$, $n = 4, \dots, 15$, we fix a reduced word w_x in letters y_1, y_2 and y_3 that represents x in $G_n/Z(G_n)$.
   $w_x = y_2 y_3$ for $C = B_3$
2. From the reduced word w_x one can obtain a word $\tilde{w_x}$ not necessarily reduced, that also represents x in $G_n/Z(G_n)$, such that
   - $\tilde{w_1} = 1$.
   - If w_x is reduced, then $\tilde{w_x} = w_x$.

The proof of BMR for the first two families

1. For every $x \in G_n/Z(G_n)$, $n = 4, \ldots, 15$, we fix a reduced word w_x in letters y_1, y_2 and y_3 that represents x in $G_n/Z(G_n)$.
   $w_x = y_2y_3$ for $C = B_3$
2. From the reduced word w_x one can obtain a word $\tilde{w}_x$ not necessarily reduced, that also represents x in $G_n/Z(G_n)$, such that
   - $\tilde{w}_1 = 1$.
   - If $\tilde{w}_x$ is reduced, then $\tilde{w}_x = w_x$.
   - Let m an odd number. Whenever in the word $\tilde{w}_x$ there is a letter y_i at the mth-position from left to right, then in the $(m+1)$th-position there is a letter y_j, $j \neq i$.

The proof of BMR for the first two families

1. For every $x \in G_n/Z(G_n)$, $n = 4, \dots, 15$, we fix a reduced word w_x in letters y_1, y_2 and y_3 that represents x in $G_n/Z(G_n)$.
   $w_x = y_2y_3$ for $C = B_3$

2. From the reduced word w_x one can obtain a word $\tilde{w}_x$ not necessarily reduced, that also represents x in $G_n/Z(G_n)$, such that

   - $\tilde{w}_1 = 1$.
   - If $\tilde{w}_x$ is reduced, then $\tilde{w}_x = w_x$.
   - Let m an odd number. Whenever in the word $\tilde{w}_x$ there is a letter y_i at the mth-position from left to right, then in the $(m+1)$th-position there is a letter y_j, $j \neq i$.
   - $\tilde{w}_x = y_2y_1y_1y_2y_2y_1y_1y_3$

The proof of BMR for the first two families

1. For every $x \in G_n/Z(G_n)$, $n = 4, \ldots, 15$, we fix a reduced word w_x in letters y_1, y_2 and y_3 that represents x in $G_n/Z(G_n)$.
   $w_x = y_2y_3$ for $C = B_3$

2. From the reduced word w_x one can obtain a word $\tilde{w}_x$ not necessarily reduced, that also represents x in $G_n/Z(G_n)$, such that
   - $\tilde{w}_1 = 1$.
   - If $\tilde{w}_x$ is reduced, then $\tilde{w}_x = w_x$.
   - Let m an odd number. Whenever in the word $\tilde{w}_x$ there is a letter y_i at the mth-position from left to right, then in the $(m+1)$th-position there is a letter y_j, $j \neq i$.
   - $\tilde{w}_x = y_2y_1y_1y_2y_2y_1y_1y_3$ $T_{\tilde{w}_x} = A_{21}A_{12}A_{21}A_{13}$

The proof of BMR for the first two families

1. For every $x \in G_n/Z(G_n)$, $n = 4, \ldots, 15$, we fix a reduced word w_x in letters y_1, y_2 and y_3 that represents x in $G_n/Z(G_n)$.
   $w_x = y_2 y_3$ for $C = B_3$
2. From the reduced word w_x one can obtain a word $\tilde{w}_x$ not necessarily reduced, that also represents x in $G_n/Z(G_n)$, such that
   - $\tilde{w}_1 = 1$.
   - If $\tilde{w}_x$ is reduced, then $\tilde{w}_x = w_x$.
   - Let m an odd number. Whenever in the word $\tilde{w}_x$ there is a letter y_i at the mth-position from left to right, then in the $(m+1)$th-position there is a letter y_j, $j \neq i$.
   - $\tilde{w}_x = y_2 y_1 y_1 y_2 y_2 y_1 y_1 y_3 \qquad T_{\tilde{w}_x} = A_{21} A_{12} A_{21} A_{13}$
3. We set $B_n = \{z^k \ (T_{\tilde{w}_x});\ k = 0, \ldots, Z(G_n) \ \ 1\}$.

The proof of BMR for the first two families

1. For every $x \in G_n/Z(G_n)$, $n = 4, \ldots, 15$, we fix a reduced word w_x in letters y_1, y_2 and y_3 that represents x in $G_n/Z(G_n)$.
   $w_x = y_2 y_3$ for $C = B_3$
2. From the reduced word w_x one can obtain a word $\tilde{w}_x$ not necessarily reduced, that also represents x in $G_n/Z(G_n)$, such that
   - $\tilde{w}_1 = 1$.
   - If $\tilde{w}_x$ is reduced, then $\tilde{w}_x = w_x$.
   - Let m an odd number. Whenever in the word $\tilde{w}_x$ there is a letter y_i at the mth-position from left to right, then in the $(m+1)$th-position there is a letter y_j, $j \neq i$.
   - $\tilde{w}_x = y_2 y_1 y_1 y_2 y_2 y_1 y_1 y_3 \qquad T_{\tilde{w}_x} = A_{21} A_{12} A_{21} A_{13}$
3. We set $B_n = \{z^k (T_{\tilde{w}_x});\ k = 0, \ldots, Z(G_n) - 1\}$.
4. We prove B_n is a spanning set of $H(G_n)$ over R_{G_n}.

The proof of BMR for the first two families

1. For every $x \in G_n/Z(G_n)$, $n = 4, \ldots, 15$, we fix a reduced word w_x in letters y_1, y_2 and y_3 that represents x in $G_n/Z(G_n)$.
   $w_x = y_2 y_3$ for $C = B_3$
2. From the reduced word w_x one can obtain a word $\tilde{w}_x$ not necessarily reduced, that also represents x in $G_n/Z(G_n)$, such that

   $\tilde{w}_1 = 1$.
   If $\tilde{w}_x$ is reduced, then $\tilde{w}_x = w_x$.
   Let m an odd number. Whenever in the word $\tilde{w}_x$ there is a letter y_i at the mth-position from left to right, then in the $(m+1)$th-position there is a letter y_j, $j \neq i$.
   $\tilde{w}_x = y_2 y_1 y_1 y_2 y_2 y_1 y_1 y_3$ $\quad T_{\tilde{w}_x} = A_{21} A_{12} A_{21} A_{13}$
3. We set $B_n = \{z^k (T_{\tilde{w}_x});\ k = 0, \ldots, Z(G_n) - 1\}$.
4. We prove B_n is a spanning set of $H(G_n)$ over R_{G_n}.

The choice of $\tilde{w}_x$ is a product of experimentation!

Last slide

Thank you!