On Hecke algebras

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Representation theory

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$$\mathrm{Irr}(\mathcal{A}) := \text{ set of irreducible representations of } \mathcal{A}$$

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Coxeter-like presentation

$$\langle s_1, \ldots, s_n \mid s_i^{o(s_i)} = 1,\ \text{homogeneous positive relations} \rangle.$$

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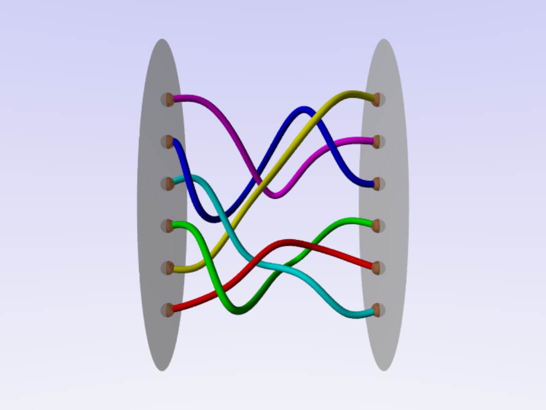
- the groups of the infinite series $G(de, e, n)$, with d, e, n positive integers and $(de, e, n) \neq (2, 2, 2)$.
- the exceptional groups $G_4, \ldots, G_{37}$.

Braid groups

Symmetric group S_n

Braid group
on n strands



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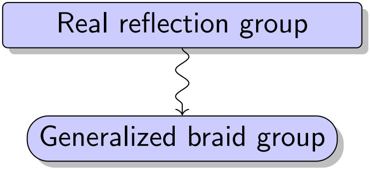

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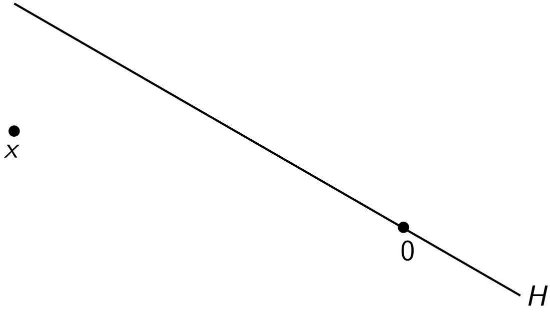

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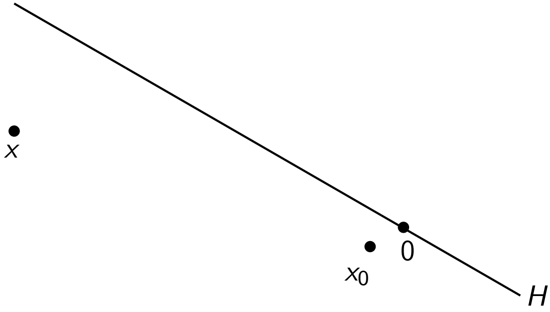

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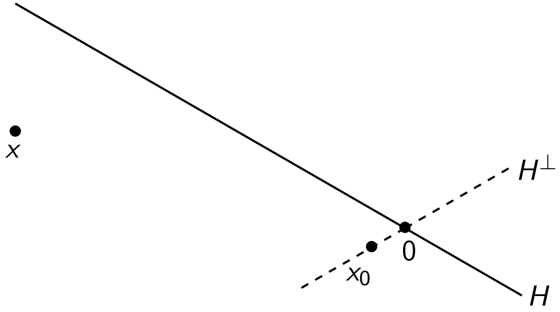

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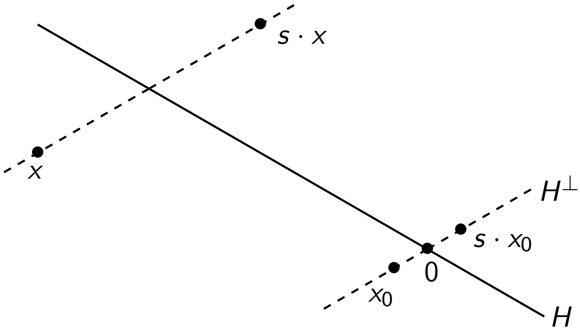

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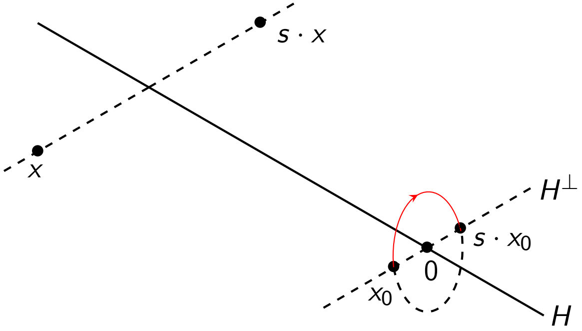

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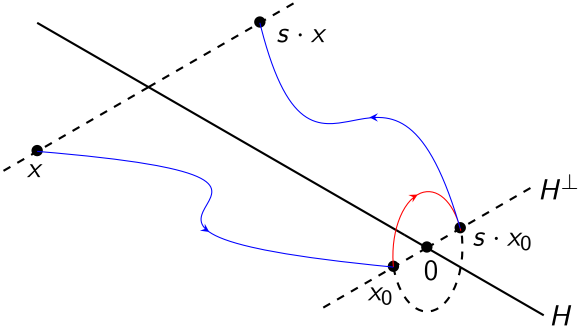

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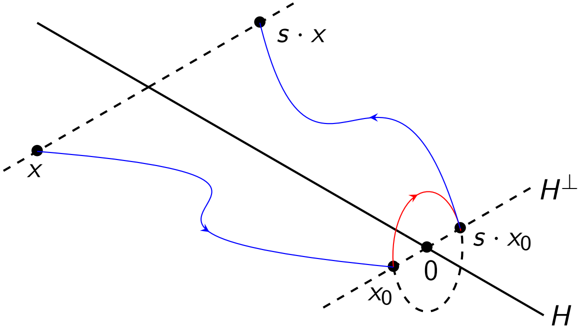

The complex braid group B is generated by distinguished braided reflections, whose image inside W are the distinguished pseudo-reflections that generate W.

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Spets $\rightsquigarrow$ complex reflection group $\rightsquigarrow$ generic Hecke algebra

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Conjecture [Broué-Malle-Rouquier 1998]

H_W admits an R-basis with $|W|$ elements.

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We provided a nice basis: it has an inductive nature and it involves powers of a central element.

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$\rightsquigarrow$ Programming (C++ and SAGE).

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Let $\theta : \mathbb{C}(\mathbf{u}) \to \mathbb{C}$ a ring homomorphism, called specialization.

Representation theory

Let W be a complex reflection group and let H_W the associated Hecke algebra, defined over $R = \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-\mathbf{1}}]$.

Theorem (Malle 1999)

The algebra $\mathbb{C}(\mathbf{u})H_W := \mathbb{C}(\mathbf{u}) \otimes_R H_W$ is split semisimple.

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If $\mathbb{C}H_k$ is semisimple, we can use Tits' deformation theorem again.

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What happens if $\mathbb{C}H_W$ is not semisimple?

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Theorem [C. 2017]

There is an optimal basic set for G_4, G_8, G_{16} with respect to any θ.

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Theorem [C.-Pfeiffer 2019]

- A nice description of $Z(H_{G_n})$, $n = 4, 5, 6, 7, 8$.
- There is always an optimal basic set for G_n, $n = 4, 5, 6, 7, 8$.

Last slide

Thank you!