Decomposition matrices for the generic Hecke algebras on 3 strands

Eirini Chavli

Universität Stuttgart

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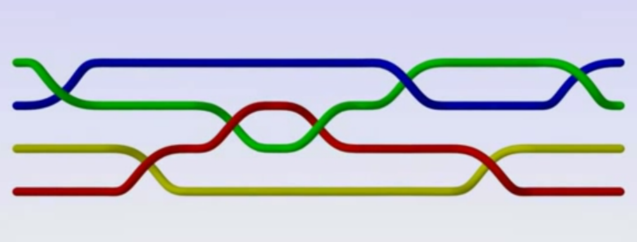
The braid group on n strands



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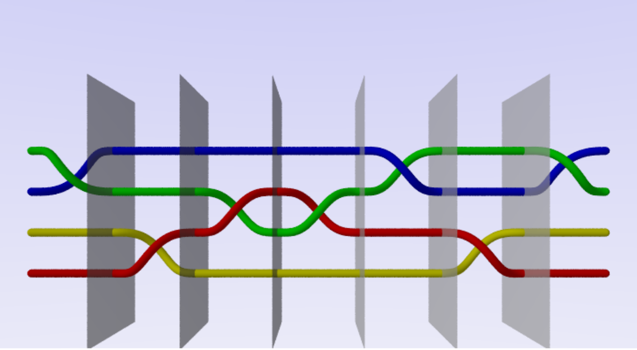

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The braid group B_n on n strands is a group that admits a presentation with generators

$$\sigma_1, \dots, \sigma_{n-1}$$

and relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \text{ for } i = 1, \dots, n-1$$

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The first "new" example is $B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle$.

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Theorem (Coxeter 1957)

The quotient W_k of the B_n by the relations $\sigma_i^k = 1$ is a finite group if and only if $\frac{1}{k} + \frac{1}{n} > \frac{1}{2}$.

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- $n = 2$. Then, $W_k = \mathbb{Z}/k\mathbb{Z}$.
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- $n = 3$ and $k = 3, 4, 5$.
- $n = 4$ and $k = 3$.
- $n = 5$ and $k = 3$.

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The generic Hecke algebra on 3 strands

Let W_k ($k = 3, 4, 5$) be a finite quotient of the braid group B_3 and $R_k := \mathbb{Z}[a_1^{\pm}, a_2^{\pm} ..., a_k^{\pm}]$.

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Conjecture 2 (Broué-Malle-Rouquier 1998)

H_k is a symmetric algebra.

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Theorem (Broué-Malle, Berceanu-Funar 1994, Marin 2011)

Conjecture 1 is true for W_3.

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Theorem (Boura-C.-Chlouveraki-Karvounis 2017)

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Assumption

Conjecture 2 is also true for W_5.

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Let $\theta : \ F_k \to \mathbb{C}$ be a specialization of $F_k H_k$ defined by $v_i \mapsto \lambda_i$.

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What happens if $\mathbb{C}H_k$ is not semisimple?

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$$R_0^+(F_k H_k) \xrightarrow{d_\theta} R_0^+(\mathbb{C} H_k)$$

The decomposition matrix

The corresponding decomposition matrix is the $\mathrm{Irr}(F_k H_k)\times$ $\mathrm{Irr}(\mathbb{C}H_k)$ matrix $(d_{\chi\phi})$ with non-negative integer entries such that $d_\theta([V_\chi]) = \sum_\phi d_{\chi\phi}[V'_\phi]$.

$$\begin{array}{c} \\ \vdots \\ \chi_i \\ \vdots \end{array} \begin{array}{c} \phi_1 \quad \phi_2 \quad \dots \quad \phi_n \\ \left(\begin{array}{cccc} & & & \\ & & & \\ & & & \end{array} \right) \end{array}$$

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This matrix records in which way the irreducible representations of the semisimple algebra $F_k H_k$ break up into irreducible representations of $\mathbb{C}H_k$.

Classification of the decomposition matrices

Theorem (Chlouveraki-Miyachi 2012)

For the Cyclotomic Hecke algebra in d-Harish Chandra Series the decomposition matrix is of the form:

$$\left(\begin{array}{ccccccc} & & & I_m & & & \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right)$$

where the entries of the low part are 0 and 1.

Classification of the decomposition matrices

Theorem (C 2017)

For the generic Hecke algebras on 3 strands the decomposition matrix is of the form:

$$\left(\begin{array}{ccccccc} & & & I_m & & & \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right)$$

where the entries of the low part are 0, 1 and 2.[a]

[a] For $k = 5$ there is one decomposition matrix still in progress.

Thank you!