The BMR freeness conjecture

Eirini Chavli

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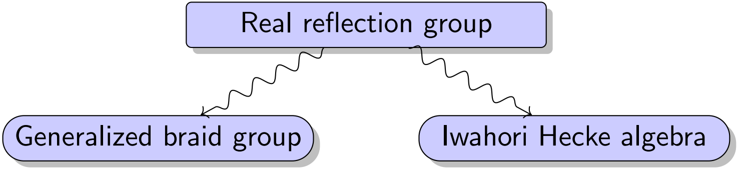
29 August 2016

From real to complex



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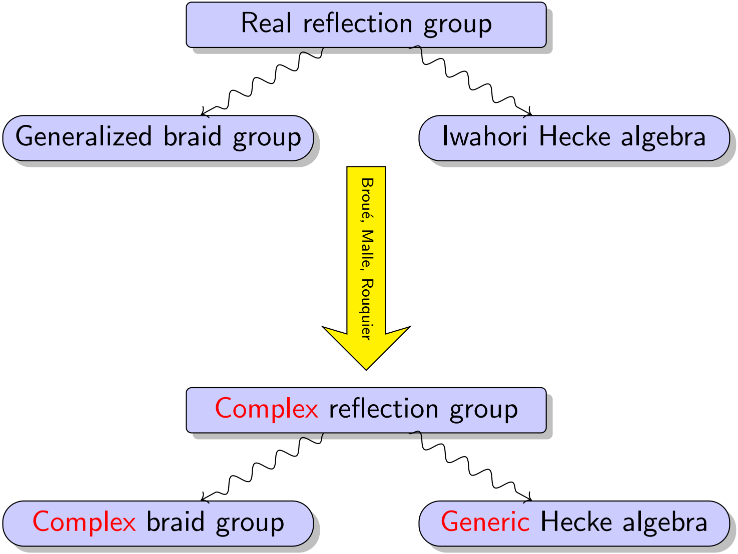

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Coxeter-like presentation: $\langle s_1, \dots, s_n \mid s_i^{o(s_i)} = 1, \{v_i = w_i\}_{i \in I} \rangle$, where I is a finite set of relations such that, for each $i \in I$, v_i and w_i are positive words with the same length in elements $s_1, \dots, s_n$.

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- For any finite Coxeter group B is a generalized braid group.

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Conjecture (equivalent form)

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Among them there are 6 finite Coxeter groups.

We have some recent results from Marin and Pfeiffer (2015).

Marin has proved the cases of G_{25}, G_{26} and G_{32} (2011, 2012).

Classification of irreducible complex reflection groups

The finite irreducible complex reflection groups were classified by Shephard and Todd (1954). They belond either to

- The infinite family $G(de, e, n)$ of groups of $n \times n$ monomials such that their non-zero coefficients belong to $\mu_{de}(\mathbb{C})$ and their product to $\mu_d(\mathbb{C})$. ✓ (Ariki, Ariki-Koike 1993)
- A finite set of 34 exceptions.

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The BMR freeness conjecture holds for the generic Hecke algebras associated to G_4[1], G_8 and G_{16}.

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&\quad + u_1 z^4 + u_1 z^{-4} + u_1 z^{-5} + u_1 u_2 u_1 + u_1 \sigma_2^{-1} \sigma_1^2 \sigma_2^{-1} u_1 + \\
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Proposition (C. 2013)

If H_W is torsion free and the BMR conjecture is true for $H_{\widetilde{W}}$, then the conjecture holds for H_W, as well.

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We know that $\overline{W}$ is the group of even elements in a finite Coxeter group C of rank 3 of type A_3, B_3, H_3, respectively, with Coxeter matrix (m_{ij}).

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Example: When C is the finite Coxeter group of type B_3 the $\tilde{R}$-algebra $A_+(C)$ can be presented as follows:

A more global approach: Etingof-Rains 2004

We know that $\overline{W}$ is the group of even elements in a finite Coxeter group C of rank 3 of type A_3, B_3, H_3, respectively, with Coxeter matrix (m_{ij}).

Let $t_{ij,k}$ be set of indeterminates, where $i \neq j$, $k \in \mathbb{Z}/m_{ij}\mathbb{Z}$ and $t_{ij,k}^{-1} = t_{ji,-k}$. Let $\tilde{R} := \mathbb{Z}[t_{ij,k}]$.

$$A(C) = \langle\ Y_1, Y_2, Y_3 \mid Y_i^2 = 1,\ i \neq j \text{ implies } \prod_{k=1}^{m_{ij}} (Y_i Y_j - t_{ij,k}) = 0\ \rangle$$

$$A_+(C) = \langle A_{ij} := Y_i Y_j, i \neq j \rangle$$

<u>Example</u>: When C is the finite Coxeter group of type B_3 the $\tilde{R}$-algebra $\overline{A_+(C)}$ can be presented as follows:

$$\left\langle\ A_{13}, A_{32}, A_{21} \;\middle|\; \begin{array}{c} (A_{13} - t_{13,1})(A_{13} - t_{13,2}) = 0 \\ (A_{32} - t_{32,1}) \ldots (A_{32} - t_{32,3}) = 0, \quad A_{13}A_{32}A_{21} = 1 \\ (A_{21} - t_{21,1}) \ldots (A_{21} - t_{21,4}) = 0 \end{array} \right\rangle$$

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Proposition (Etingof-Rains 2004)

There is a ring morphism $\phi : \tilde{R} \twoheadrightarrow R^+$, inducing $\phi : A_+(C) \otimes_\theta R^+ \twoheadrightarrow H_W$, considering H_W as R^+-module.

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- $\ell(\tilde{w}_x) = \ell(\bar{w}_x)$, where $\ell(w)$ denotes the length of the word w.
- Let m be an odd number. Whenever in the word $\tilde{w}_x$ there is a letter y_i at the mth-position from left to right, then in the $(m+1)$th-position there is a letter y_j, $j \neq i$.
- $\tilde{w}_x = w_x$ if and only if $\bar{w}_x = w_x$. In particular, $\tilde{w}_1 = w_1$.

$\tilde{w}_x = y_2 \mathbf{y_1 y_1 y_2 y_2 y_1 y_1} y_3$

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$$U := \sum_{x \in \overline{W}} \sum_{k=0}^{|Z(W)|-1} R z^k T_{\tilde{w}_x},$$

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Theorem (C. 2015)

$H_W = U$, and, hence, the BMR freeness conjecture is true for the groups belonging to the first two families.

Remaining cases

Conjecture (equivalent version)

H_W is spanned over R by $|W|$ elements, where W is an irreducible complex reflection group.

$$\underbrace{G_4, \ldots, G_7}_{\mathcal{T}}, \underbrace{G_8, \ldots, G_{15}}_{\mathcal{O}}, \underbrace{G_{16}, G_{17}, G_{18}, G_{19}, G_{20}, G_{21}, G_{22}}_{\mathcal{I}}, \quad G_{23}, G_{24}, \ldots G_{36}, G_{37}.$$

Last slide

Thank you!