Modular representation theory for the generic Hecke algebras on 3 strands.

Eirini Chavli

Universität Stuttgart

21 June 2018

Introduction

Let G be a finite group and K a field of any characteristic.

Introduction

Let G be a finite group and K a field of any characteristic.

- Representation theory of $G \rightsquigarrow KG$-modules.

Introduction

Let G be a finite group and K a field of any characteristic.

- Representation theory of $G \rightsquigarrow KG$-modules.
- We would like to "replace" KG with another algebra A, keeping also some nice properties of KG.

Introduction

Let G be a finite group and K a field of any characteristic.

- Representation theory of $G \rightsquigarrow KG$-modules.
- We would like to "replace" KG with another algebra A, keeping also some nice properties of KG.

  A is a finitely generated symmetric algebra.

Introduction

Let G be a finite group and K a field of any characteristic.

- Representation theory of $G \rightsquigarrow KG$-modules.
- We would like to "replace" KG with another algebra A, keeping also some nice properties of KG.

  A is a finitely generated symmetric algebra.
- Ordinary representation theory (A is a split semisimple algebra).
- Modular representation theory (A is not a split semisimple algebra).

Introduction

Let G be a finite group and K a field of any characteristic.

- Representation theory of $G \rightsquigarrow KG$-modules.
- We would like to "replace" KG with another algebra A, keeping also some nice properties of KG.

  A is a finitely generated symmetric algebra.
- Ordinary representation theory (A is a split semisimple algebra).
- Modular representation theory (A is not a split semisimple algebra).
  $\rightsquigarrow$ The theory of decomposition maps.

The braid group on n strands

The braid group on n strands

The braid group on n strands

Let $n \in \mathbb{N}$.

The braid group on n strands

Let $n \in \mathbb{N}$.

Theorem (Artin 1928)

The braid group B_n on n strands is a group that admits a presentation with generators

$$\sigma_1, \ldots, \sigma_{n-1}$$

and relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \text{ for } i = 1, \ldots, n-1$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \text{ for } |i-j| > 1$$

The braid group on n strands

Let $n \in \mathbb{N}$.

Theorem (Artin 1928)

The braid group B_n on n strands is a group that admits a presentation with generators

$$\sigma_1, \ldots, \sigma_{n-1}$$

and relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \text{ for } i = 1, \ldots, n-1$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \text{ for } |i - j| > 1$$

- For $n = 1$ we have $B_1 = 1$.

The braid group on n strands

Let $n \in \mathbb{N}$.

Theorem (Artin 1928)

The braid group B_n on n strands is a group that admits a presentation with generators

$$\sigma_1, \ldots, \sigma_{n-1}$$

and relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \text{ for } i = 1, \ldots, n-1$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \text{ for } |i-j| > 1$$

- For $n = 1$ we have $B_1 = 1$.
- For $n = 2$ we have $B_2 = \mathbb{Z}$.

The braid group on n strands

Let $n \in \mathbb{N}$.

Theorem (Artin 1928)

The braid group B_n on n strands is a group that admits a presentation with generators

$$\sigma_1, \ldots, \sigma_{n-1}$$

and relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \text{ for } i = 1, \ldots, n-1$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \text{ for } |i - j| > 1$$

- For $n = 1$ we have $B_1 = 1$.
- For $n = 2$ we have $B_2 = \mathbb{Z}$.

The first "new" example is $B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle$.

The braid group B_3

- B_3 is the knot group of the trefoil knot

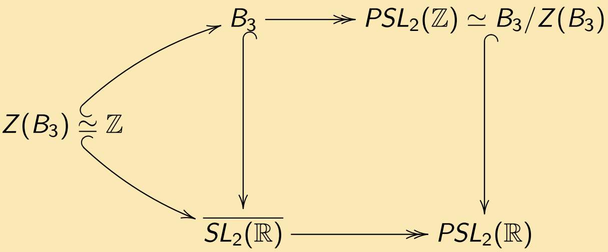
$$B_3 \simeq \pi_1 \left(\mathbb{R}^3 / \qquad \right)$$

The braid group B_3

- B_3 is the knot group of the trefoil knot

$$B_3 \simeq \pi_1 \left(\mathbb{R}^3 / \qquad \right)$$

- B_3 is the universal central extension of the modular group $PSL_2(\mathbb{Z})$

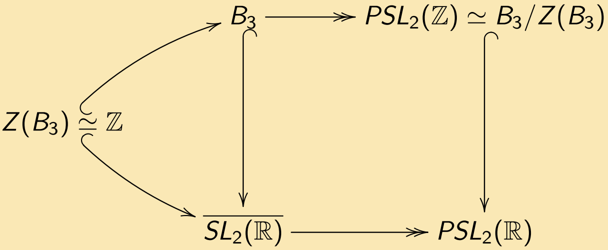


The braid group B_3

- B_3 is the knot group of the trefoil knot

$$B_3 \simeq \pi_1\left(\mathbb{R}^3/ \qquad \right)$$

- B_3 is the universal central extension of the modular group $PSL_2(\mathbb{Z})$



- B_3 is connected with cubic invariants, including the Kaufman polynomial and the Links-Gould polynomial.

The finite quotients of B_n

$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle.$

The finite quotients of B_n

$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$. We have $S_n = B_n/(\sigma_i^2)$.

The finite quotients of B_n

$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$. We have $S_n = B_n/(\sigma_i^2)$.

Question

When the quotient $W_k := B_n/(\sigma_i^k)$ is finite?

The finite quotients of B_n

$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$. We have $S_n = B_n/(\sigma_i^2)$.

Question

When the quotient $W_k := B_n/(\sigma_i^k)$ is finite?

Theorem (Coxeter 1957)

The quotient W_k of the B_n by the relations $\sigma_i^k = 1$ is a finite group if and only if $\frac{1}{k} + \frac{1}{n} > \frac{1}{2}$.

The finite quotients of B_n

$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$. We have $S_n = B_n/(\sigma_i^2)$.

Question

When the quotient $W_k := B_n/(\sigma_i^k)$ is finite?

Theorem (Coxeter 1957)

The quotient W_k of the B_n by the relations $\sigma_i^k = 1$ is a finite group if and only if $\frac{1}{k} + \frac{1}{n} > \frac{1}{2}$.

- $n = 1$. Then $W_k = \{1\}$.
- $n = 2$. Then, $W_k = \mathbb{Z}/k\mathbb{Z}$.
- $k = 2$. Then, $W_2 = S_n$.

The finite quotients of B_n

$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$. We have $S_n = B_n/(\sigma_i^2)$.

Question

When the quotient $W_k := B_n/(\sigma_i^k)$ is finite?

Theorem (Coxeter 1957)

The quotient W_k of the B_n by the relations $\sigma_i^k = 1$ is a finite group if and only if $\frac{1}{k} + \frac{1}{n} > \frac{1}{2}$.

- $n = 1$. Then $W_k = \{1\}$.
- $n = 2$. Then, $W_k = \mathbb{Z}/k\mathbb{Z}$.
- $k = 2$. Then, $W_2 = S_n$.
- $n = 3$ and $k = 3, 4, 5$.
- $n = 4$ and $k = 3$.
- $n = 5$ and $k = 3$.

The finite quotients of B_n

$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$. We have $S_n = B_n/(\sigma_i^2)$.

Question

When the quotient $W_k := B_n/(\sigma_i^k)$ is finite?

Theorem (Coxeter 1957)

The quotient W_k of the B_n by the relations $\sigma_i^k = 1$ is a finite group if and only if $\frac{1}{k} + \frac{1}{n} > \frac{1}{2}$.

- $n = 1$. Then $W_k = \{1\}$.
- $n = 2$. Then, $W_k = \mathbb{Z}/k\mathbb{Z}$.
- $k = 2$. Then, $W_2 = S_n$.
- $n = 3$ and $k = 3, 4, 5$.
- $n = 4$ and $k = 3$.
- $n = 5$ and $k = 3$.

The finite quotients of B_n

$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$. We have $S_n = B_n/(\sigma_i^2)$.

Question

When the quotient $W_k := B_n/(\sigma_i^k)$ is finite?

Theorem (Coxeter 1957)

The quotient W_k of the B_n by the relations $\sigma_i^k = 1$ is a finite group if and only if $\frac{1}{k} + \frac{1}{n} > \frac{1}{2}$.

- $n = 1$. Then $W_k = \{1\}$.
- $n = 2$. Then, $W_k = \mathbb{Z}/k\mathbb{Z}$.
- $k = 2$. Then, $W_2 = S_n$.
- $n = 3$ and $k = 3, 4, 5$. $W_3 := G_4$, $W_4 := G_8$, $W_5 := G_{16}$.
- $n = 4$ and $k = 3$.
- $n = 5$ and $k = 3$.

The generic Hecke algebras on 3 strands

Definition

The generic Hecke algebra H_k of W_k on 3 strands is defined as $H_k := R_k B_3/(\sigma_i - a_1)...(\sigma_i - a_k)$.

The generic Hecke algebras on 3 strands

Definition

The generic Hecke algebra H_k of W_k on 3 strands is defined as $H_k := R_k B_3/(\sigma_i - a_1)...(\sigma_i - a_k)$. It is a deformation of $R_k W_k$.

The generic Hecke algebras on 3 strands

Definition

The generic Hecke algebra H_k of W_k on 3 strands is defined as $H_k := R_k B_3/(\sigma_i - a_1)...(\sigma_i - a_k)$. It is a deformation of $R_k W_k$.

Example:

- $W_3 = \langle s_1, s_2 \mid s_1 s_2 s_1 = s_2 s_1 s_2,\ s_i^3 = 1 \rangle$.

The generic Hecke algebras on 3 strands

Definition

The generic Hecke algebra H_k of W_k on 3 strands is defined as $H_k := R_k B_3/(\sigma_i - a_1)...(\sigma_i - a_k)$. It is a deformation of $R_k W_k$.

Example:

- $W_3 = \langle s_1, s_2 \mid s_1 s_2 s_1 = s_2 s_1 s_2,\ s_i^3 = 1 \rangle$.
- $H_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2,\ (\sigma_i - a_1)(\sigma_i - a_2)(\sigma_i - a_3) = 0 \rangle$.

The generic Hecke algebras on 3 strands

Definition

The generic Hecke algebra H_k of W_k on 3 strands is defined as $H_k := R_k B_3/(\sigma_i - a_1)...(\sigma_i - a_k)$. It is a deformation of $R_k W_k$.

Example:

- $W_3 = \langle s_1, s_2 \mid s_1 s_2 s_1 = s_2 s_1 s_2,\ s_i^3 = 1 \rangle$.
- $H_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2,\ (\sigma_i - a_1)(\sigma_i - a_2)(\sigma_i - a_3) = 0 \rangle$.

Conjecture 1 (Broué-Malle-Rouquier 1998)

H_k is a free R_k-module of dimension $|W_k|$.

The generic Hecke algebras on 3 strands

Definition

The generic Hecke algebra H_k of W_k on 3 strands is defined as $H_k := R_k B_3/(\sigma_i - a_1)...(\sigma_i - a_k)$. It is a deformation of $R_k W_k$.

Example:

- $W_3 = \langle s_1, s_2 \mid s_1 s_2 s_1 = s_2 s_1 s_2,\ s_i^3 = 1 \rangle$.
- $H_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2,\ (\sigma_i - a_1)(\sigma_i - a_2)(\sigma_i - a_3) = 0 \rangle$.

Conjecture 1 (Broué-Malle-Rouquier 1998)

H_k is a free R_k-module of dimension $|W_k|$.

Conjecture 2 (Broué-Malle-Michel 1999)

H_k is symmetric

The generic Hecke algebras on 3 strands

Definition

The generic Hecke algebra H_k of W_k on 3 strands is defined as $H_k := R_k B_3/(\sigma_i - a_1)...(\sigma_i - a_k)$. It is a deformation of $R_k W_k$.

Example:

- $W_3 = \langle s_1, s_2 \mid s_1 s_2 s_1 = s_2 s_1 s_2,\ s_i^3 = 1 \rangle$.
- $H_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2,\ (\sigma_i - a_1)(\sigma_i - a_2)(\sigma_i - a_3) = 0 \rangle$.

Conjecture 1 (Broué-Malle-Rouquier 1998)

H_k is a free R_k-module of dimension $|W_k|$.

Conjecture 2 (Broué-Malle-Michel 1999)

H_k is symmetric i.e. there is a linear map $t:\ H_k \to R_k$ such that:

- $t(h_1 h_2) = t(h_2 h_1)$, for all $h_1, h_2 \in H_k$.
- The bilinear form $(h_1, h_2) \mapsto t(h_1 h_2)$ is non-degenerate.

The generic Hecke algebras on 3 strands

Conjecture 1 is now a theorem.

The generic Hecke algebras on 3 strands

Conjecture 1 is now a theorem. For the finite quotients of B_3 we have:

- The validity of Conjecture 1 for W_3 by Broué-Malle, Berceanu-Funar 1994, Marin 2011, C. 2017.
- The validity of Conjecture 1 for W_4 and W_5 by C. 2016.

The generic Hecke algebras on 3 strands

Conjecture 1 is now a theorem. For the finite quotients of B_3 we have:

- The validity of Conjecture 1 for W_3 by Broué-Malle, Berceanu-Funar 1994, Marin 2011, C. 2017.
- The validity of Conjecture 1 for W_4 and W_5 by C. 2016.

Considering now Conjecture 2, we have:

- Conjecture 2 is true for W_3 (Malle-Michel 2010, Marin-Wagner 2012, Boura-C.-Chlouveraki-Karvounis 2018).
- Conjecture 2 is true for W_4 (Boura-C.-Chlouveraki-Karvounis 2018).

The generic Hecke algebras on 3 strands

Conjecture 1 is now a theorem. For the finite quotients of B_3 we have:

- The validity of Conjecture 1 for W_3 by Broué-Malle, Berceanu-Funar 1994, Marin 2011, C. 2017.
- The validity of Conjecture 1 for W_4 and W_5 by C. 2016.

Considering now Conjecture 2, we have:

- Conjecture 2 is true for W_3 (Malle-Michel 2010, Marin-Wagner 2012, Boura-C.-Chlouveraki-Karvounis 2018).
- Conjecture 2 is true for W_4 (Boura-C.-Chlouveraki-Karvounis 2018).

Assumption

Conjecture 2 is also true for W_5.

The idea of the proof for Conjecture 1

Let W_1 be the subgroup of G_n $(n = 4, 8, 16)$ which is generated by s_1.

The idea of the proof for Conjecture 1

Let W_1 be the subgroup of G_n $(n = 4, 8, 16)$ which is generated by s_1.
Let $\mathcal{J}$ denote a system of representatives of the double cosets $W_1 \backslash G_n / W_1$.

The idea of the proof for Conjecture 1

Let W_1 be the subgroup of G_n $(n = 4, 8, 16)$ which is generated by s_1.
Let $\mathcal{J}$ denote a system of representatives of the double cosets $W_1 \backslash G_n / W_1$.

$$G_n = \bigsqcup_{w \in \mathcal{J}} W_1 \cdot w \cdot W_1.$$

The idea of the proof for Conjecture 1

Let W_1 be the subgroup of G_n ($n = 4, 8, 16$) which is generated by s_1.
Let $\mathcal{J}$ denote a system of representatives of the double cosets $W_1 \backslash G_n / W_1$.

$$G_n = \bigsqcup_{w \in \mathcal{J}} W_1 \cdot w \cdot W_1.$$

For every $w \in \mathcal{J}$ we fix a factorization $f_w = s_{i_1}^{a_1} s_{i_2}^{a_2} \ldots s_{i_r}^{a_r}$, where $a_i \in \mathbb{Z}$.

The idea of the proof for Conjecture 1

Let W_1 be the subgroup of G_n $(n = 4, 8, 16)$ which is generated by s_1.
Let $\mathcal{J}$ denote a system of representatives of the double cosets $W_1 \backslash G_n / W_1$.

$$G_n = \bigsqcup_{w \in \mathcal{J}} W_1 \cdot w \cdot W_1.$$

For every $w \in \mathcal{J}$ we fix a factorization $f_w = s_{i_1}^{a_1} s_{i_2}^{a_2} \dots s_{i_r}^{a_r}$, where $a_i \in \mathbb{Z}$.
We define the element T_{f_w} inside H_{G_n} to be the product $\sigma_{i_1}^{a_1} \sigma_{i_2}^{a_2} \dots \sigma_{i_r}^{a_r}$ inside H_k $(T_{f_1} = 1)$.

The idea of the proof for Conjecture 1

Let W_1 be the subgroup of G_n $(n = 4, 8, 16)$ which is generated by s_1.
Let $\mathcal{J}$ denote a system of representatives of the double cosets $W_1 \backslash G_n / W_1$.

$$G_n = \bigsqcup_{w \in \mathcal{J}} W_1 \cdot w \cdot W_1.$$

For every $w \in \mathcal{J}$ we fix a factorization $f_w = s_{i_1}^{a_1} s_{i_2}^{a_2} \dots s_{i_r}^{a_r}$, where $a_i \in \mathbb{Z}$.
We define the element T_{f_w} inside H_{G_n} to be the product $\sigma_{i_1}^{a_1} \sigma_{i_2}^{a_2} \dots \sigma_{i_r}^{a_r}$ inside H_k $(T_{f_1} = 1)$.
Let u_1 be the subalgebra of H_{G_n} $(n = 4, 8, 16)$ which is generated by σ_1.

The idea of the proof for Conjecture 1

Let W_1 be the subgroup of G_n $(n = 4, 8, 16)$ which is generated by s_1.
Let $\mathcal{J}$ denote a system of representatives of the double cosets $W_1 \backslash G_n / W_1$.

$$G_n = \bigsqcup_{w \in \mathcal{J}} W_1 \cdot w \cdot W_1.$$

For every $w \in \mathcal{J}$ we fix a factorization $f_w = s_{i_1}^{a_1} s_{i_2}^{a_2} \dots s_{i_r}^{a_r}$, where $a_i \in \mathbb{Z}$.
We define the element T_{f_w} inside H_{G_n} to be the product $\sigma_{i_1}^{a_1} \sigma_{i_2}^{a_2} \dots \sigma_{i_r}^{a_r}$ inside H_k $(T_{f_1} = 1)$.
Let u_1 be the subalgebra of H_{G_n} $(n = 4, 8, 16)$ which is generated by σ_1.

$$H_{G_n} \stackrel{?}{=} \sum_{w \in \mathcal{J}} u_1 \cdot T_{f_w} \cdot u_1$$

The idea of the proof for Conjecture 1

Let W_1 be the subgroup of G_n $(n = 4, 8, 16)$ which is generated by s_1. Let $\mathcal{J}$ denote a system of representatives of the double cosets $W_1 \backslash W_k / W_1$.

$$G_n = \bigsqcup_{w \in \mathcal{J}} W_1 \cdot w \cdot W_1.$$

For every $w \in \mathcal{J}$ we fix a factorization $f_w = s_{i_1}^{a_1} s_{i_2}^{a_2} \dots s_{i_r}^{a_r}$, where $a_i \in \mathbb{Z}$. We define the element T_{f_w} inside H_{G_n} to be the product $\sigma_{i_1}^{a_1} \sigma_{i_2}^{a_2} \dots \sigma_{i_r}^{a_r}$ inside H_k $(T_{f_1} = 1)$.
Let u_1 be the subalgebra of H_{G_n} $(n = 4, 8, 16)$ which is generated by σ_1.

$$H_{G_n} \stackrel{?}{=} \underbrace{\sum_{w \in \mathcal{J}} u_1 \cdot T_{f_w} \cdot u_1}_{U}$$

The idea of the proof for Conjecture 1

Let W_1 be the subgroup of G_n ($n = 4, 8, 16$) which is generated by s_1. Let $\mathcal{J}$ denote a system of representatives of the double cosets $W_1 \backslash W_k / W_1$.

$$G_n = \bigsqcup_{w \in \mathcal{J}} W_1 \cdot w \cdot W_1.$$

For every $w \in \mathcal{J}$ we fix a factorization $f_w = s_{i_1}^{a_1} s_{i_2}^{a_2} \dots s_{i_r}^{a_r}$, where $a_i \in \mathbb{Z}$. We define the element T_{f_w} inside H_{G_n} to be the product $\sigma_{i_1}^{a_1} \sigma_{i_2}^{a_2} \dots \sigma_{i_r}^{a_r}$ inside H_k ($T_{f_1} = 1$).
Let u_1 be the subalgebra of H_{G_n} ($n = 4, 8, 16$) which is generated by σ_1.

$$H_{G_n} \overset{?}{=} \underbrace{\sum_{w \in \mathcal{J}} u_1 \cdot T_{f_w} \cdot u_1}_{U}$$

Since $1 \in U$, it is enough to prove that $\sigma_i U \subset U$

The idea of the proof for Conjecture 1

Let W_1 be the subgroup of G_n ($n = 4, 8, 16$) which is generated by s_1.
Let $\mathcal{J}$ denote a system of representatives of the double cosets $W_1 \backslash W_k / W_1$.

$$G_n = \bigsqcup_{w \in \mathcal{J}} W_1 \cdot w \cdot W_1.$$

For every $w \in \mathcal{J}$ we fix a factorization $f_w = s_{i_1}^{a_1} s_{i_2}^{a_2} \dots s_{i_r}^{a_r}$, where $a_i \in \mathbb{Z}$.
We define the element T_{f_w} inside H_{G_n} to be the product $\sigma_{i_1}^{a_1} \sigma_{i_2}^{a_2} \dots \sigma_{i_r}^{a_r}$ inside H_k ($T_{f_1} = 1$).
Let u_1 be the subalgebra of H_{G_n} ($n = 4, 8, 16$) which is generated by σ_1.

$$H_{G_n} \stackrel{?}{=} \underbrace{\sum_{w \in \mathcal{J}} u_1 \cdot T_{f_w} \cdot u_1}_{U}$$

Since $1 \in U$, it is enough to prove that $\sigma_i U \subset U$ i.e. it is enough to prove that $\sigma_2 U \subset U$.

The irreducible representations of B_3 of dimension $k \leq 5$

The irreducible representations of B_3 of dimension $k \leq 5$

Let $\rho : \; B_3 \to \; GL_k(\mathbb{C})$ a representation of B_3 of dimension $k \leq 5$ over $\mathbb{C}$.

Let $A := \rho(\sigma_1)$ and $B := \rho(\sigma_2)$, with eigenvalues $\lambda_1, \ldots, \lambda_k$.

The irreducible representations of B_3 of dimension $k \leq 5$

Let $\rho : B_3 \to GL_k(\mathbb{C})$ a representation of B_3 of dimension $k \leq 5$ over $\mathbb{C}$.

Let $A := \rho(\sigma_1)$ and $B := \rho(\sigma_2)$, with eigenvalues $\lambda_1, \ldots, \lambda_k$.

By the theorem of Cayley-Hamilton there is a monic polynomial $Q(X) = (X - \lambda_1) \ldots (X - \lambda_k) \in \mathbb{C}[X]$ such that $Q(A) = Q(B) = 0$.

The irreducible representations of B_3 of dimension $k \leq 5$

Let $\rho : \; B_3 \to \; GL_k(\mathbb{C})$ a representation of B_3 of dimension $k \leq 5$ over $\mathbb{C}$.

Let $A := \rho(\sigma_1)$ and $B := \rho(\sigma_2)$, with eigenvalues $\lambda_1, \ldots, \lambda_k$.

By the theorem of Cayley-Hamilton there is a monic polynomial $Q(X) = (X - \lambda_1) \ldots (X - \lambda_k) \in \mathbb{C}[X]$ such that $Q(A) = Q(B) = 0$.

$\rho(\sigma_1)\rho(\sigma_2)\rho(\sigma_1) = \rho(\sigma_2)\rho(\sigma_1)\rho(\sigma_2)$,

The irreducible representations of B_3 of dimension $k \leq 5$

Let $\rho : \ B_3 \to \ GL_k(\mathbb{C})$ a representation of B_3 of dimension $k \leq 5$ over $\mathbb{C}$.

Let $A := \rho(\sigma_1)$ and $B := \rho(\sigma_2)$, with eigenvalues $\lambda_1, \ldots, \lambda_k$.

By the theorem of Cayley-Hamilton there is a monic polynomial $Q(X) = (X - \lambda_1) \ldots (X - \lambda_k) \in \mathbb{C}[X]$ such that $Q(A) = Q(B) = 0$.

$\rho(\sigma_1)\rho(\sigma_2)\rho(\sigma_1) = \rho(\sigma_2)\rho(\sigma_1)\rho(\sigma_2), \quad (\rho(\sigma_i) - \lambda_1 \mathrm{I}_k) \ldots (\rho(\sigma_i) - \lambda_k \mathrm{I}_k) = 0.$

The irreducible representations of B_3 of dimension $k \leq 5$

Let $\rho: \ B_3 \to \ GL_k(\mathbb{C})$ a representation of B_3 of dimension $k \leq 5$ over $\mathbb{C}$.

Let $A := \rho(\sigma_1)$ and $B := \rho(\sigma_2)$, with eigenvalues $\lambda_1, \ldots, \lambda_k$.

By the theorem of Cayley-Hamilton there is a monic polynomial $Q(X) = (X - \lambda_1)\ldots(X - \lambda_k) \in \mathbb{C}[X]$ such that $Q(A) = Q(B) = 0$.

$\rho(\sigma_1)\rho(\sigma_2)\rho(\sigma_1) = \rho(\sigma_2)\rho(\sigma_1)\rho(\sigma_2), \quad (\rho(\sigma_i) - \lambda_1 \mathrm{I}_k)\ldots(\rho(\sigma_i) - \lambda_k \mathrm{I}_k) = 0.$

Recall that $H_k = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, (\sigma_i - a_1)\ldots(\sigma_i - a_k) = 0 \rangle$ defined over $R_k := \mathbb{Z}[a_1^{\pm}, ..., a_k^{\pm}]$.

The irreducible representations of B_3 of dimension $k \leq 5$

Let $\rho: \; B_3 \to \; GL_k(\mathbb{C})$ a representation of B_3 of dimension $k \leq 5$ over $\mathbb{C}$.

Let $A := \rho(\sigma_1)$ and $B := \rho(\sigma_2)$, with eigenvalues $\lambda_1, \dots, \lambda_k$.

By the theorem of Cayley-Hamilton there is a monic polynomial $Q(X) = (X - \lambda_1)\dots(X - \lambda_k) \in \mathbb{C}[X]$ such that $Q(A) = Q(B) = 0$.

$\rho(\sigma_1)\rho(\sigma_2)\rho(\sigma_1) = \rho(\sigma_2)\rho(\sigma_1)\rho(\sigma_2), \quad (\rho(\sigma_i) - \lambda_1 \mathrm{I}_k)\dots(\rho(\sigma_i) - \lambda_k \mathrm{I}_k) = 0.$

Recall that $H_k = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, (\sigma_i - a_1)\dots(\sigma_i - a_k) = 0 \rangle$ defined over $R_k := \mathbb{Z}[a_1^{\pm}, \dots, a_k^{\pm}]$.

Let $\theta: \; R_k \to \mathbb{C}$ a ring homomorphism, defined by $a_i \mapsto \lambda_i$ ($\lambda_i \neq 0$).

The irreducible representations of B_3 of dimension $k \leq 5$

Let $\rho : B_3 \to GL_k(\mathbb{C})$ a representation of B_3 of dimension $k \leq 5$ over $\mathbb{C}$.

Let $A := \rho(\sigma_1)$ and $B := \rho(\sigma_2)$, with eigenvalues $\lambda_1, \dots, \lambda_k$.

By the theorem of Cayley-Hamilton there is a monic polynomial $Q(X) = (X - \lambda_1)\dots(X - \lambda_k) \in \mathbb{C}[X]$ such that $Q(A) = Q(B) = 0$.

$\rho(\sigma_1)\rho(\sigma_2)\rho(\sigma_1) = \rho(\sigma_2)\rho(\sigma_1)\rho(\sigma_2), \quad (\rho(\sigma_i) - \lambda_1 \mathrm{I}_k)\dots(\rho(\sigma_i) - \lambda_k \mathrm{I}_k) = 0.$

Recall that $H_k = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, (\sigma_i - a_1)\dots(\sigma_i - a_k) = 0 \rangle$ defined over $R_k := \mathbb{Z}[a_1^{\pm}, \dots, a_k^{\pm}]$.

Let $\theta : R_k \to \mathbb{C}$ a ring homomorphism, defined by $a_i \mapsto \lambda_i$ ($\lambda_i \neq 0$).

We only need to describe the irreducible $\mathbb{C}H_k := H_k \otimes_\theta \mathbb{C}$-modules of dimension k.

The irreducible representations of B_3 of dimension $k \leq 5$

Let $\rho : \ B_3 \to \ GL_k(\mathbb{C})$ a representation of B_3 of dimension $k \leq 5$ over $\mathbb{C}$.

Let $A := \rho(\sigma_1)$ and $B := \rho(\sigma_2)$, with eigenvalues $\lambda_1, \ldots, \lambda_k$.

By the theorem of Cayley-Hamilton there is a monic polynomial $Q(X) = (X - \lambda_1) \ldots (X - \lambda_k) \in \mathbb{C}[X]$ such that $Q(A) = Q(B) = 0$.

$\rho(\sigma_1)\rho(\sigma_2)\rho(\sigma_1) = \rho(\sigma_2)\rho(\sigma_1)\rho(\sigma_2), \quad (\rho(\sigma_i) - \lambda_1 \mathrm{I}_k) \ldots (\rho(\sigma_i) - \lambda_k \mathrm{I}_k) = 0.$

Recall that $H_k = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, (\sigma_i - a_1) \ldots (\sigma_i - a_k) = 0 \rangle$ defined over $R_k := \mathbb{Z}[a_1^{\pm}, \ldots, a_k^{\pm}]$.

Let $\theta : \ R_k \to \mathbb{C}$ a ring homomorphism, defined by $a_i \mapsto \lambda_i$ ($\lambda_i \neq 0$).

We only need to describe the irreducible $\mathbb{C}H_k := H_k \otimes_\theta \mathbb{C}$-modules of dimension k.

Theorem (C. 2016)

We can classify the irreducible representations of B_3 of dimension at most 5 over $\mathbb{C}$.

The decomposition map

Theorem (Malle 1999)

There is a field $F_k \supset R_k$ such that $F_k H_k := H_k \otimes_{R_k} F_k$ is split semisimple.

The decomposition map

Theorem (Malle 1999)

There is a field $F_k \supset R_k$ such that $F_k H_k := H_k \otimes_{R_k} F_k$ is split semisimple.

$F_k := \mathbb{C}(v_1, \ldots, v_k)$, where each parameter v_i is an N_i-th root of a_i for some specific $N_i \in \mathbb{N}$.

The decomposition map

Theorem (Malle 1999)

There is a field $F_k \supset R_k$ such that $F_k H_k := H_k \otimes_{R_k} F_k$ is split semisimple.

$F_k := \mathbb{C}(v_1, \ldots, v_k)$, where each parameter v_i is an N_i-th root of a_i for some specific $N_i \in \mathbb{N}$.

Theorem (Geck 1990)

$$t_k = \sum_{\chi \in Irr(F_k H_k)} \frac{1}{s_\chi} \chi,$$

The decomposition map

Theorem (Malle 1999)

There is a field $F_k \supset R_k$ such that $F_k H_k := H_k \otimes_{R_k} F_k$ is split semisimple.

$F_k := \mathbb{C}(v_1, \ldots, v_k)$, where each parameter v_i is an N_i-th root of a_i for some specific $N_i \in \mathbb{N}$.

Theorem (Geck 1990)

$$t_k = \sum_{\chi \in Irr(F_k H_k)} \frac{1}{s_\chi} \chi,$$

where s_χ is the Schur element of χ with respect to t_k.

The decomposition map

Theorem (Malle 1999)

There is a field $F_k \supset R_k$ such that $F_k H_k := H_k \otimes_{R_k} F_k$ is split semisimple.

$F_k := \mathbb{C}(v_1, \ldots, v_k)$, where each parameter v_i is an N_i-th root of a_i for some specific $N_i \in \mathbb{N}$.

Theorem (Geck 1990)

$$t_k = \sum_{\chi \in Irr(F_k H_k)} \frac{1}{s_\chi} \chi,$$

where s_χ is the Schur element of χ with respect to t_k.

By Tits' deformation theorem the specialization $v_i \mapsto 1$ induces a bijection $\mathrm{Irr}(F_k H_k) \to \mathrm{Irr}(W_k)$.

The decomposition map

Theorem (Malle 1999)

There is a field $F_k \supset R_k$ such that $F_k H_k := H_k \otimes_{R_k} F_k$ is split semisimple.

$F_k := \mathbb{C}(v_1, \ldots, v_k)$, where each parameter v_i is an N_i-th root of a_i for some specific $N_i \in \mathbb{N}$.

Theorem (Geck 1990)

$$t_k = \sum_{\chi \in Irr(F_k H_k)} \frac{1}{s_\chi} \chi,$$

where s_χ is the Schur element of χ with respect to t_k.

By Tits' deformation theorem the specialization $v_i \mapsto 1$ induces a bijection $\mathrm{Irr}(F_k H_k) \to \mathrm{Irr}(W_k)$.
Let $\theta : R_k \to \mathbb{C}$ be a specialization of H_k defined by $a_i \mapsto \lambda_i$.

The decomposition map

Theorem (Malle 1999)

There is a field $F_k \supset R_k$ such that $F_k H_k := H_k \otimes_{R_k} F_k$ is split semisimple.

$F_k := \mathbb{C}(v_1, \ldots, v_k)$, where each parameter v_i is an N_i-th root of a_i for some specific $N_i \in \mathbb{N}$.

Theorem (Geck 1990)

$$t_k = \sum_{\chi \in Irr(F_k H_k)} \frac{1}{s_\chi} \chi,$$

where s_χ is the Schur element of χ with respect to t_k.

By Tits' deformation theorem the specialization $v_i \mapsto 1$ induces a bijection $\mathrm{Irr}(F_k H_k) \to \mathrm{Irr}(W_k)$.
Let $\theta : \ R_k \to \mathbb{C}$ be a specialization of H_k defined by $a_i \mapsto \lambda_i$.
The algebra $\mathbb{C}H_k := H_k \otimes_\theta \mathbb{C}$ is split.

The decomposition map

Theorem (Malle 1999)

There is a field $F_k \supset R_k$ such that $F_k H_k := H_k \otimes_{R_k} F_k$ is split semisimple.

$F_k := \mathbb{C}(v_1, \ldots, v_k)$, where each parameter v_i is an N_i-th root of a_i for some specific $N_i \in \mathbb{N}$.

Theorem (Geck 1990)

$$t_k = \sum_{\chi \in Irr(F_k H_k)} \frac{1}{s_\chi} \chi,$$

where s_χ is the Schur element of χ with respect to t_k.

By Tits' deformation theorem the specialization $v_i \mapsto 1$ induces a bijection $\mathrm{Irr}(F_k H_k) \to \mathrm{Irr}(W_k)$.

Let $\theta : \ R_k \to \mathbb{C}$ be a specialization of H_k defined by $a_i \mapsto \lambda_i$.

The algebra $\mathbb{C}H_k := H_k \otimes_\theta \mathbb{C}$ is split. If $\mathbb{C}H_k$ is semisimple, we can use Tits' deformation theorem again.

The decomposition map

What happens if $\mathbb{C}H_k$ is not semisimple?

The decomposition map

What happens if $\mathbb{C}H_k$ is not semisimple?

Let $R_0(F_k H_k)$ be the Grothendieck group of the category of finite-dimensional $F_k H_k$-modules i.e.

The decomposition map

What happens if $\mathbb{C}H_k$ is not semisimple?

Let $R_0(F_k H_k)$ be the Grothendieck group of the category of finite-dimensional $F_k H_k$-modules i.e.
$R_0(F_k H_k)$ is generated by expressions $[V]$, with relations $[V] = [V'] + [V'']$ for each exact sequence $0 \to V' \to V \to V'' \to 0$.

The decomposition map

What happens if $\mathbb{C}H_k$ is not semisimple?

Let $R_0(F_kH_k)$ be the Grothendieck group of the category of finite-dimensional F_kH_k-modules i.e.
$R_0(F_kH_k)$ is generated by expressions $[V]$, with relations $[V] = [V'] + [V'']$ for each exact sequence $0 \to V' \to V \to V'' \to 0$.

Let $R_0^+(F_kH_k)$ be the subset of $R_0(F_kH_k)$ consisting of elements $[V]$, where V is a finite-dimensional F_kH_k-module.

The decomposition map

What happens if $\mathbb{C}H_k$ is not semisimple?

Let $R_0(F_k H_k)$ be the Grothendieck group of the category of finite-dimensional $F_k H_k$-modules i.e.
$R_0(F_k H_k)$ is generated by expressions $[V]$, with relations $[V] = [V'] + [V'']$ for each exact sequence $0 \to V' \to V \to V'' \to 0$.

Let $R_0^+(F_k H_k)$ be the subset of $R_0(F_k H_k)$ consisting of elements $[V]$, where V is a finite-dimensional $F_k H_k$-module.

We can define the decomposition map d^θ associated to θ.

The decomposition map

What happens if $\mathbb{C}H_k$ is not semisimple?

Let $R_0(F_k H_k)$ be the Grothendieck group of the category of finite-dimensional $F_k H_k$-modules i.e.
$R_0(F_k H_k)$ is generated by expressions $[V]$, with relations $[V] = [V'] + [V'']$ for each exact sequence $0 \to V' \to V \to V'' \to 0$.

Let $R_0^+(F_k H_k)$ be the subset of $R_0(F_k H_k)$ consisting of elements $[V]$, where V is a finite-dimensional $F_k H_k$-module.

We can define the decomposition map d^θ associated to θ.

$$R_0^+(F_k H_k) \xrightarrow{d^\theta} R_0^+(\mathbb{C}H_k)$$

The decomposition map

$$R_0^+(F_k H_k) \xrightarrow{d^\theta} R_0^+(\mathbb{C} H_k)$$

The decomposition map

$$R_0^+(F_k H_k) \xrightarrow{d^\theta} R_0^+(\mathbb{C} H_k)$$

Let V_χ be an irreducible $F_k H_k$-module with character χ.

We set $[M] := d^\theta([V_\chi])$ and we suppose that M is not irreducible.

The decomposition map

$$R_0^+(F_k H_k) \xrightarrow{d^\theta} R_0^+(\mathbb{C} H_k)$$

Let V_χ be an irreducible $F_k H_k$-module with character χ.

We set $[M] := d^\theta([V_\chi])$ and we suppose that M is not irreducible. M admits an irreducible submodule S_M, which gives rise to the exact sequence $0 \to S_M \to M \to M/S_M \to 0$.

The decomposition map

$$R_0^+(F_k H_k) \xrightarrow{d^\theta} R_0^+(\mathbb{C}H_k)$$

Let V_χ be an irreducible $F_k H_k$-module with character χ.

We set $[M] := d^\theta([V_\chi])$ and we suppose that M is not irreducible. M admits an irreducible submodule S_M, which gives rise to the exact sequence $0 \to S_M \to M \to M/S_M \to 0$.

We have $[M] = [S_M] + [M/S_M]$.

The decomposition map

$$R_0^+(F_k H_k) \xrightarrow{d^\theta} R_0^+(\mathbb{C}H_k)$$

Let V_χ be an irreducible $F_k H_k$-module with character χ.

We set $[M] := d^\theta([V_\chi])$ and we suppose that M is not irreducible. M admits an irreducible submodule S_M, which gives rise to the exact sequence $0 \to S_M \to M \to M/S_M \to 0$.

We have $[M] = [S_M] + [M/S_M]$.

If the module M/S_M is not irreducible, we have a new exact sequence and we repeat the procedure.

The decomposition map

$$R_0^+(F_k H_k) \xrightarrow{d^\theta} R_0^+(\mathbb{C}H_k)$$

Let V_χ be an irreducible $F_k H_k$-module with character χ.

We set $[M] := d^\theta([V_\chi])$ and we suppose that M is not irreducible. M admits an irreducible submodule S_M, which gives rise to the exact sequence $0 \to S_M \to M \to M/S_M \to 0$.

We have $[M] = [S_M] + [M/S_M]$.

If the module M/S_M is not irreducible, we have a new exact sequence and we repeat the procedure. This procedure terminates, since the dimension of the quotient module is strictly decreasing, and every 1-dimensional module is irreducible.

The decomposition map

$$R_0^+(F_kH_k) \xrightarrow{d^\theta} R_0^+(\mathbb{C}H_k)$$

Let V_χ be an irreducible F_kH_k-module with character χ.

We set $[M] := d^\theta([V_\chi])$ and we suppose that M is not irreducible. M admits an irreducible submodule S_M, which gives rise to the exact sequence $0 \to S_M \to M \to M/S_M \to 0$.

We have $[M] = [S_M] + [M/S_M]$.

If the module M/S_M is not irreducible, we have a new exact sequence and we repeat the procedure. This procedure terminates, since the dimension of the quotient module is strictly decreasing, and every 1-dimensional module is irreducible.

We finally obtain $d^\theta([V_\chi]) = \sum_\phi d^\theta_{\chi\phi}[V'_\phi]$, where $d^\theta_{\chi\phi}$ are non–negative integers and V'_ϕ denotes the irreducible $\mathbb{C}H_{W_k}$-module with character ϕ.

The decomposition matrix

The corresponding decomposition matrix is the $\mathrm{Irr}(F_k H_k) \times \mathrm{Irr}(\mathbb{C} H_k)$ matrix $(d^{\theta}_{\chi\phi})$.

The decomposition matrix

The corresponding decomposition matrix is the $\mathrm{Irr}(F_k H_k) \times \mathrm{Irr}(\mathbb{C} H_k)$ matrix $(d^{\theta}_{\chi\phi})$.

$$\begin{array}{c} \\ \vdots \\ \chi_i \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} \phi_1 & \phi_2 & \dots & \phi_n \end{array} \\ \left(\begin{array}{cccc} & & & \\ & & & \\ & & & \end{array} \right) \end{array}$$

The decomposition matrix

The corresponding decomposition matrix is the $\mathrm{Irr}(F_k H_k) \times \mathrm{Irr}(\mathbb{C}H_k)$ matrix $(d^{\theta}_{\chi\phi})$.

$$\begin{array}{c} \\ \vdots \\ \chi_i \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} \phi_1 & \phi_2 & \cdots & \phi_n \end{array} \\ \left(\begin{array}{cccc} & & & \\ & & & \\ & & & \end{array} \right) \end{array}$$

$$d_{\theta}([V_{\chi_i}]) = d^{\theta}_{\chi,\phi_1}[V'_{\phi_1}] + d^{\theta}_{\chi,\phi_2}[V'_{\phi_2}] + \cdots + d^{\theta}_{\chi,\phi_n}[V'_{\phi_n}].$$

The decomposition matrix

The corresponding decomposition matrix is the $\mathrm{Irr}(F_k H_k)\times$ $\mathrm{Irr}(\mathbb{C}H_k)$ matrix $(d^{\theta}_{\chi\phi})$.

$$\begin{array}{c} \\ \vdots \\ \chi_i \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccc} \phi_1 & \phi_2 & \cdots & \phi_n \end{array} \\ \left(\begin{array}{cccc} & & & \\ d^{\theta}_{\chi,\phi_1} & d^{\theta}_{\chi,\phi_2} & \cdots & d^{\theta}_{\chi,\phi_n} \\ & & & \end{array} \right) \end{array}$$

$$d_{\theta}([V_{\chi_i}]) = d^{\theta}_{\chi,\phi_1}[V'_{\phi_1}] + d^{\theta}_{\chi,\phi_2}[V'_{\phi_2}] + \cdots + d^{\theta}_{\chi,\phi_n}[V'_{\phi_n}].$$

The decomposition matrix

The corresponding decomposition matrix is the $\mathrm{Irr}(F_k H_k) \times \mathrm{Irr}(\mathbb{C}H_k)$ matrix $(d^{\theta}_{\chi\phi})$.

$$\begin{array}{c c} & \begin{array}{cccc} \phi_1 & \phi_2 & \cdots & \phi_n \end{array} \\ \begin{array}{c} \vdots \\ \chi_i \\ \vdots \end{array} & \begin{pmatrix} \\ d^{\theta}_{\chi,\phi_1} & d^{\theta}_{\chi,\phi_2} & \cdots & d^{\theta}_{\chi,\phi_n} \\ \\ \end{pmatrix} \end{array}$$

$$d_{\theta}([V_{\chi_i}]) = d^{\theta}_{\chi,\phi_1}[V'_{\phi_1}] + d^{\theta}_{\chi,\phi_2}[V'_{\phi_2}] + \cdots + d^{\theta}_{\chi,\phi_n}[V'_{\phi_n}].$$

This matrix depicts the way the irreducible representations of the semisimple algebra $F_k H_k$ break up into irreducible representations of $\mathbb{C}H_k$.

The decomposition matrix

We say that the $F_k H_k$-modules $V_\chi, V_{\chi'}$ belong to the same block if $\chi \neq \chi'$ and there is a chain of characters $\chi = \chi_1, \chi_2, \ldots, \chi_n = \chi'$, where for every two neighbors χ_i, χ_{i+1} there is a character $\phi_i \in \mathrm{Irr}(\mathbb{C}H_{W_k})$ such that $d^\theta_{\chi_i,\phi_i} \neq 0 \neq d^\theta_{\chi_{i+1},\phi_i}$.

The decomposition matrix

We say that the $F_k H_k$-modules V_χ, $V_{\chi'}$ belong to the same block if $\chi \neq \chi'$ and there is a chain of characters $\chi = \chi_1, \chi_2, \ldots, \chi_n = \chi'$, where for every two neighbors χ_i, χ_{i+1} there is a character $\phi_i \in \mathrm{Irr}(\mathbb{C}H_{W_k})$ such that $d^\theta_{\chi_i,\phi_i} \neq 0 \neq d^\theta_{\chi_{i+1},\phi_i}$.

$$\begin{array}{c} \\ \chi_i \\ \chi_{i+1} \end{array} \begin{array}{c} \begin{array}{ccc} \ldots & \phi_i & \ldots \end{array} \\ \begin{pmatrix} * & a & * \\ * & b & * \end{pmatrix} \end{array}$$

where $ab \neq 0$.

The decomposition matrix

We say that the $F_k H_k$-modules V_χ, $V_{\chi'}$ belong to the same block if $\chi \neq \chi'$ and there is a chain of characters $\chi = \chi_1, \chi_2, \ldots, \chi_n = \chi'$, where for every two neighbors χ_i, χ_{i+1} there is a character $\phi_i \in \mathrm{Irr}(\mathbb{C}H_{W_k})$ such that $d^\theta_{\chi_i,\phi_i} \neq 0 \neq d^\theta_{\chi_{i+1},\phi_i}$.

$$\begin{array}{cc} & \begin{array}{ccc} \ldots & \phi_i & \ldots \end{array} \\ \begin{array}{c} \chi_i \\ \chi_{i+1} \end{array} & \left(\begin{array}{ccc} * & a & * \\ * & b & * \end{array} \right) \end{array}$$

where $ab \neq 0$.

If an irreducible $F_k H_k$-module is alone in its block, then we call it a module of defect 0.

The decomposition matrix

We say that the $F_k H_k$-modules $V_\chi, V_{\chi'}$ belong to the same block if $\chi \neq \chi'$ and there is a chain of characters $\chi = \chi_1, \chi_2, \ldots, \chi_n = \chi'$, where for every two neighbors χ_i, χ_{i+1} there is a character $\phi_i \in \mathrm{Irr}(\mathbb{C}H_{W_k})$ such that $d^\theta_{\chi_i,\phi_i} \neq 0 \neq d^\theta_{\chi_{i+1},\phi_i}$.

$$\begin{array}{c} \\ \chi_i \\ \chi_{i+1} \end{array} \begin{array}{c} \begin{array}{ccc} \ldots & \phi_i & \ldots \end{array} \\ \begin{pmatrix} * & a & * \\ * & b & * \end{pmatrix} \end{array}$$

where $ab \neq 0$.

If an irreducible $F_k H_k$-module is alone in its block, then we call it a module of defect 0.

- We have $\theta(s_\chi) \neq 0$ if and only if V_χ is a module of defect 0 (Geck-Pfeiffer 1999).

The decomposition matrix

We say that the $F_k H_k$-modules V_χ, $V_{\chi'}$ belong to the same block if $\chi \neq \chi'$ and there is a chain of characters $\chi = \chi_1, \chi_2, \ldots, \chi_n = \chi'$, where for every two neighbors χ_i, χ_{i+1} there is a character $\phi_i \in \mathrm{Irr}(\mathbb{C}H_{W_k})$ such that $d^\theta_{\chi_i,\phi_i} \neq 0 \neq d^\theta_{\chi_{i+1},\phi_i}$.

$$\begin{array}{c} \\ \chi_i \\ \chi_{i+1} \end{array} \begin{array}{c} \begin{array}{ccc} \cdots & \phi_i & \cdots \end{array} \\ \left(\begin{array}{ccc} * & a & * \\ * & b & * \end{array}\right) \end{array}$$

where $ab \neq 0$.

If an irreducible $F_k H_k$-module is alone in its block, then we call it a module of defect 0.

- We have $\theta(s_\chi) \neq 0$ if and only if V_χ is a module of defect 0 (Geck-Pfeiffer 1999).
- If V_χ, $V_{\chi'}$ are in the same block, then $\theta(\omega_\chi(z_0)) = \theta(\omega_{\chi'}(z_0))$, where $\omega_\chi, \omega_{\chi'}$ are the corresponding central characters and z_0 is the central element $(\sigma_1\sigma_2)^3$ (Geck-Pfeiffer 1999).

Questions

Let S be an irreducible k-dimensional $\mathbb{C}H_k$-module.

Questions

Let S be an irreducible k-dimensional $\mathbb{C}H_k$-module.

Proposition (C. 2016)

There is an irreducible $F_K H_k$-module M such that $d^\theta([M]) = [S] + [J]$, where J is an $\mathbb{C}H_k$-module.

Questions

Let S be an irreducible k-dimensional $\mathbb{C}H_k$-module.

Proposition (C. 2016)

There is an irreducible $F_K H_k$-module M such that $d^\theta([M]) = [S] + [J]$, where J is an $\mathbb{C}H_k$-module.

- Does this hold for every irreducible $\mathbb{C}H_k$-module?

Questions

Let S be an irreducible k-dimensional $\mathbb{C}H_k$-module.

Proposition (C. 2016)

There is an irreducible $F_K H_k$-module M such that $d^\theta([M]) = [S] + [J]$, where J is an $\mathbb{C}H_k$-module.

- Does this hold for every irreducible $\mathbb{C}H_k$-module?
- Can we describe J?

Questions

Let S be an irreducible k-dimensional $\mathbb{C}H_k$-module.

Proposition (C. 2016)

There is an irreducible $F_K H_k$-module M such that $d^\theta([M]) = [S] + [J]$, where J is an $\mathbb{C}H_k$-module.

- Does this hold for every irreducible $\mathbb{C}H_k$-module?
- Can we describe J?
- Is there an irreducible $F_K H_k$-module M' such that $d^\theta([M']) = [S]$?

Classification of the decomposition matrices

Theorem (Chlouveraki-Miyachi 2012)

For the Cyclotomic Hecke algebra in d-Harish Chandra Series[a] the decomposition matrix is of the form:

$$\left(\begin{array}{ccc} & I_m & \\ \hline \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}\right)$$

where $\cdot$ denotes a placeholder for one of two values: 0 or 1.

[a] **Example** $H_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2,\ (\sigma_i - 1)(\sigma_i - q)(\sigma_i - q^2) = 0 \rangle$

Classification of the decomposition matrices

Theorem (Chlouveraki-Miyachi 2012)

For the Cyclotomic Hecke algebra in d-Harish Chandra Series[a] the decomposition matrix is of the form:

$$\left(\begin{array}{ccc} & I_m & \\ \hline \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}\right)$$

where $\cdot$ denotes a placeholder for one of two values: 0 or 1.

[a] **Example** $H_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2,\ (\sigma_i - 1)(\sigma_i - q)(\sigma_i - q^2) = 0 \rangle$

The upper part of the matrix is indexed by a subset of $\mathrm{Irr}(W_k)$, which is called optimal basic set.

Classification of the decomposition matrices

Theorem (C. 2017)

For the generic Hecke algebras on 3 strands the decomposition matrix is of the form:

$$\left(\begin{array}{ccc} & I_m & \\ \hline \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}\right)$$

where $\cdot$ denotes a placeholder for one of three values: 0, 1 or 2.

Methodology

Methodology

Theorem (Geck-Pfeiffer 1999)

$\mathbb{C}H_k$ is semisimple if and only if $\theta(s_\chi) \neq 0$, for every $\chi \in \mathrm{Irr}(F_k H_k)$.

Methodology

Theorem (Geck-Pfeiffer 1999)

$\mathbb{C}H_k$ is semisimple if and only if $\theta(s_\chi) \neq 0$, for every $\chi \in \mathrm{Irr}(F_k H_k)$.

Theorem (Chouveraki 2007)

s_χ are products of cyclotomic polynomials evaluated on monomials of degree 0.

Methodology

Theorem (Geck-Pfeiffer 1999)

$\mathbb{C}H_k$ is semisimple if and only if $\theta(s_\chi) \neq 0$, for every $\chi \in \mathrm{Irr}(F_k H_k)$.

Theorem (Chouveraki 2007)

s_χ are products of cyclotomic polynomials evaluated on monomials of degree 0.

- We spot the characters χ of defect 0 ($\theta(s_\chi) \neq 0$).

Methodology

Theorem (Geck-Pfeiffer 1999)

$\mathbb{C}H_k$ is semisimple if and only if $\theta(s_\chi) \neq 0$, for every $\chi \in$Irr$(F_k H_k)$.

Theorem (Chouveraki 2007)

s_χ are products of cyclotomic polynomials evaluated on monomials of degree 0.

- We spot the characters χ of defect 0 ($\theta(s_\chi) \neq 0$).
- We examine the characters χ with $\theta(s_\chi) = 0$.

Methodology

Theorem (Geck-Pfeiffer 1999)

$\mathbb{C}H_k$ is semisimple if and only if $\theta(s_\chi) \neq 0$, for every $\chi \in \mathrm{Irr}(F_k H_k)$.

Theorem (Chouveraki 2007)

s_χ are products of cyclotomic polynomials evaluated on monomials of degree 0.

- We spot the characters χ of defect 0 ($\theta(s_\chi) \neq 0$).
- We examine the characters χ with $\theta(s_\chi) = 0$. We provide necessary and sufficient condition for the simplicity of every character.

Methodology

Theorem (Geck-Pfeiffer 1999)

$\mathbb{C}H_k$ is semisimple if and only if $\theta(s_\chi) \neq 0$, for every $\chi \in \mathrm{Irr}(F_k H_k)$.

Theorem (Chouveraki 2007)

s_χ are products of cyclotomic polynomials evaluated on monomials of degree 0.

- We spot the characters χ of defect 0 ($\theta(s_\chi) \neq 0$).
- We examine the characters χ with $\theta(s_\chi) = 0$. We provide necessary and sufficient condition for the simplicity of every character.
    - The character χ is simple.

Methodology

Theorem (Geck-Pfeiffer 1999)

$\mathbb{C}H_k$ is semisimple if and only if $\theta(s_\chi) \neq 0$, for every $\chi \in \mathrm{Irr}(F_k H_k)$.

Theorem (Chouveraki 2007)

s_χ are products of cyclotomic polynomials evaluated on monomials of degree 0.

- We spot the characters χ of defect 0 ($\theta(s_\chi) \neq 0$).
- We examine the characters χ with $\theta(s_\chi) = 0$. We provide necessary and sufficient condition for the simplicity of every character.
  - The character χ is simple. We check whether this character is in the same block with a character of the same dimension.

Methodology

Theorem (Geck-Pfeiffer 1999)

$\mathbb{C}H_k$ is semisimple if and only if $\theta(s_\chi) \neq 0$, for every $\chi \in \mathrm{Irr}(F_k H_k)$.

Theorem (Chouveraki 2007)

s_χ are products of cyclotomic polynomials evaluated on monomials of degree 0.

- We spot the characters χ of defect 0 ($\theta(s_\chi) \neq 0$).
- We examine the characters χ with $\theta(s_\chi) = 0$. We provide necessary and sufficient condition for the simplicity of every character.
    - The character χ is simple. We check whether this character is in the same block with a character of the same dimension. If this does not happen, we check whether χ appears in the linear combination of a non-simple character.

Methodology

Theorem (Geck-Pfeiffer 1999)

$\mathbb{C}H_k$ is semisimple if and only if $\theta(s_\chi) \neq 0$, for every $\chi \in \mathrm{Irr}(F_k H_k)$.

Theorem (Chouveraki 2007)

s_χ are products of cyclotomic polynomials evaluated on monomials of degree 0.

- We spot the characters χ of defect 0 ($\theta(s_\chi) \neq 0$).
- We examine the characters χ with $\theta(s_\chi) = 0$. We provide necessary and sufficient condition for the simplicity of every character.
  - The character χ is simple. We check whether this character is in the same block with a character of the same dimension. If this does not happen, we check whether χ appears in the linear combination of a non-simple character.
  - The character χ is not simple. We use GAP in order to write χ as a linear combination of simple characters.

Using GAP

Example

For $k = 3$ we have $W_3 = G_4$. We consider the specialization θ defined as $a_1 \mapsto a$, $a_2 \mapsto b$, $a_3 \mapsto c$. We assume $a = b$ and $a^2 - ac + c^2 = 0$.

Using GAP

Example

For $k = 3$ we have $W_3 = G_4$. We consider the specialization θ defined as $a_1 \mapsto a$, $a_2 \mapsto b$, $a_3 \mapsto c$. We assume $a = b$ and $a^2 - ac + c^2 = 0$.

```
gap> W_3:=ComplexReflectionGroup(4);;
gap> H_3:=Hecke(W_3,[[Mvp("a"), Mvp("b"), Mvp("c")]]);;
```

Using GAP

Example

For $k = 3$ we have $W_3 = G_4$. We consider the specialization θ defined as $a_1 \mapsto a$, $a_2 \mapsto b$, $a_3 \mapsto c$. We assume $a = b$ and $a^2 - ac + c^2 = 0$.

```
gap> W_3:=ComplexReflectionGroup(4);;
gap> H_3:=Hecke(W_3,[[Mvp("a"), Mvp("b"), Mvp("c")]]);;

gap> s:=SchurElements(H_3);;
gap> List(s,i->Value(i,["b", Mvp("a"), "c", -E(3)*Mvp("a")]));
gap> [0, 0, 0, 0, 0, E3, 3]
```

Using GAP

Example

For $k = 3$ we have $W_3 = G_4$. We consider the specialization θ defined as $a_1 \mapsto a$, $a_2 \mapsto b$, $a_3 \mapsto c$. We assume $a = b$ and $a^2 - ac + c^2 = 0$.

```
gap> W_3:=ComplexReflectionGroup(4);;
gap> H_3:=Hecke(W_3,[[Mvp("a"), Mvp("b"), Mvp("c")]]);;
```

```
gap> s:=SchurElements(H_3);;
gap> List(s,i->Value(i,["b", Mvp("a"), "c", -E(3)*Mvp("a")]));
gap> [0, 0, 0, 0, 0, E3, 3]
```

```
gap> T:=CharTable(W_3).irreducibles;;
gap> t:=List(T,i->List(i,j->Value(j,["b", Mvp("a"), "c",
-E(3)*Mvp("a")])));;
gap> t[1]=t[2];
true
gap> t[4]=t[5];
true
gap> t[5]=t[1]+t[3];
true
```

Last slide

Thank you!