THE CENTER OF THE WALLED BRAUER ALGEBRA $B(r,1)(\delta)$

EIRINI CHAVLI, MAUD DE VISSCHER, ALISON PARKER, SARAH SALMON, AND ULRICA WILSON

To the memory of Emmelia Kokota

Abstract. We prove that the center of the generic walled Brauer algebra $B_{r,1}(\delta)$ is generated by supersymmetric polynomials evaluated in Jucys-Murphy elements. We provide also a general strategy to obtain the same result in the case of $B_{r,s}(\delta)$.

1. Introduction

Let V be the natural representation of $\mathrm{GL}_n(V)$. There are two actions on the rth tensor product $V^{\otimes r}$: the first action is by $\mathrm{GL}_n(V)$, which is the (left) natural factor-wise action. The second one is the (right) natural action of the symmetric group S_n of permuting the tensor factors. One can easily see that these two actions compute. However, there is a stronger relation between them, as asserted by *Schur-Weyl duality* [14], namely that the span of the image of S_n and $\mathrm{GL}_n(V)$ in $\mathrm{End}(V^{\otimes r})$ are centralizers of each other.

The Brauer algebra $B_r(\delta)$ was introduced by Brauer [1], in the context of classical invariant theory, to play the role of the symmetric group in a corresponding Schur-Weyl duality for the symplectic groups (for δ positive integer) and for the orthogonal groups (for δ negative integer). The Brauer algebra can be defined for arbitrary $\delta \in \mathbb{C}$, and it is always semi-simple for $\delta \notin \mathbb{Z}$ (see, for example, [13]).

The walled Brauer algebra $B_{r,s}(\delta)$ is a subalgebra of $B_{r+s}(\delta)$ and it was introduced independently by Turaev and by Koike [6, 12]. Their motivation was the corresponding Schur-Weyl duality relating the walled Brauer algebra $B_{r,s}(n)$ with the action of $GL_n(\mathbb{C})$ on mixed tensor space $V^{\otimes r} \otimes (V^*)^{\otimes s}$.

Brauer algebras and walled Brauer algebras are particular examples of *diagram* algebras. In their diagrammatic description, one can see the group algebra of the symmetric group as their subalgebra. The goal of this paper is to give a description of the center of the walled Brauer algebra $B_{r,s}(\delta)$. Therefore, we first mention some properties of the center of the group algebra $\mathbb{C}S_r$.

The center of $\mathbb{C}S_r$ has a (natural) basis $\{C_\mu \,|\, \mu \text{ is a partition of } r\}$, where C_μ denotes the sum of all permutations with the same cycle type μ. The above elements were first introduced in [4, 8].

$$L_k := \sum_{j=1}^{k-1}(j,k), \;\; 1 \leq k \leq r$$

They are called the *Jucys-Murphy elements* of $\mathbb{C}S_r$ and they play an important role in the description of its center. More precisely, the center of $\mathbb{C}S_r$ consists of all the symmetric polynomials evaluated in Jucys-Murphy elements (for more details, see [4, 7]).

Jucys-Murphy elements for the algebra $B_{r,s}(\delta)$ were introduced in [2, 9, 10] in order to describe its center by analogy to the center of the symmetric group algebra. More precisely, it is conjectured (see [2, Remark 2.6] and [10, Conjecture 7.3]) that symmetric polynomials in [2] and some particular *doubly symmetric polynomials* in [10] evaluated in Jucys-Murphy elements generate the center of $B_{r,s}(\delta)$.

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In [5] Jung and Kim, inspired by [10, Conjecture 7.3], introduced a new family of Jucys-Murphy elements for the walled Brauer algebra $B_{r,s}(\delta)$, which we denote again as $L_1, \dots, L_{r+s}$. They proved that *supersymmetric polynomials* in these elements are central in $B_{r,s}(\delta)$. With this new definition (which is, in fact, a modification of the Jucys-Murphy elements defined in [10]) they proved the following result in the semisimple case:

Theorem 1.1. [5, Theorem 3.5] *If the walled Brauer algebra $B_{r,s}(\delta)$ is semisimple, then the supersymmetric polynomials in $L_1, \dots, L_{r+s}$ generate the center of $B_{r,s}(\delta)$.*

It is known [3, Theorem 6.3] that the algebra $B_{r,s}(\delta)$ is semisimple, except finitely many values of $\delta \in \mathbb{C}$ for a fixed pair (r, s). Moreover, in the semisimple case, the dimension of the center is the same as the number of the isomorphism classes of simple modules. Therefore, we need to recall some facts about the simple modules of the walled Brauer algebra.

For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, set $|\lambda| := \sum_{i \geq 1} \lambda_i$. Let us denote Λ the set of bipartitions and let $k \in \mathbb{N}_0$. We set:

$$\begin{aligned}
\Lambda_{r,s} &:= \coprod_{k=0}^{\min(r,s)} \{(\lambda, \mu) \in \Lambda \,|\, |\lambda| = r - k,\, |\mu| = s - k\} \\
\dot{\Lambda}_{r,s} &:= \begin{cases} \Lambda_{r,s} & \text{if } \delta \neq 0 \text{ or } r \neq s \text{ or } r = s = 0, \\ \Lambda_{r,s} - \{(\varnothing, \varnothing)\} & \text{if } \delta = 0 \text{ and } r = s \neq 0. \end{cases}
\end{aligned}$$

For $(\lambda, \mu) \in \dot{\Lambda}_{r,s}$, there is an indecomposable module $C_{r,s}((\lambda, \mu))$, called the *cell module*. Each cell module $C_{r,s}((\lambda, \mu))$ has an irreducible head $D_{r,s}((\lambda, \mu))$ and the family $\{D_{r,s}((\lambda, \mu)) \,|\, (\lambda, \mu) \in \dot{\Lambda}_{r,s}\}$ is the complete set of mutually non-isomorphic simple modules over $B_{r,s}(\delta)$ (see [3, Theorem 2.7]).

In the proof of [5, Theorem 3.5], Jung and Kim provide a set of supersymmetric polynomials $\{p_\lambda \,|\, \lambda \in \dot{\Lambda}_{r,s}\}$ such that

$$\{p_\lambda(L_1, \dots, L_{r+s}) \,|\, \lambda \in \dot{\Lambda}_{r,s}\}$$

is a basis of the center of $B_{r,s}(\delta)$.

At this point, a natural question about the generic case arises (see [5, Conjecture 5.4]). The aim of this paper is to investigate this question, namely to see if for the generic case, the center of the walled Brauer algebra $B_{r,s}(\delta)$ is generated by supersymmetric polynomials in Jucys-Murphy elements $L_1, \dots, L_{r+s}$ and if its dimension is $|\Lambda_{r,s}|$.

We will describe briefly the general strategy to obtain such a result, starting with a result of Shalile [11]. Shalile associated to each (r, s)-walled Brauer diagram d a set of words $c(d)$ called *walled generalized cycle types*, which they play the role of cycle types in the symmetric group, in the following sense ([11, Section 10]): For each walled generalized cycle type μ we set $C_\mu := \{d \in B_{r,s}(\delta) \,|\, c(d) = \mu\}$. A basis of the centralizer $Z_{B_{r,s}(\delta)}(\mathbb{C}[S_r \times S_s])$ of the subalgebra $\mathbb{C}[S_r \times S_s]$ in $B_{r,s}(\delta)$ is of the form

$$\{\sum_{d \in C_\mu} d \,|\, C_\mu \text{ walled generalized cycle type } \}$$

.

The walled Brauer algebra is generated by the generators of $S_r \times S_s$ and of an extra generator denoted by e. Hence, in order to find the center of $B_{r,s}(\delta)$, we consider the equation

$$\sum_\mu a_\mu \sum_{d \in C_\mu} (de - ed) = \sum_{x \in \mathcal{B}} b_x x.$$

Our goal is to solve the system $\{b_x = 0\}$, which can be seen as a set of linear equations in the variables a_μ. More precisely, we need to prove that the rank of this system equals

$$k := \#\text{ walled generalized cycle types} - |\Lambda_{r,s}|.$$

Using the result of Jung and Kim in the semisimple case, we prove that for any choice of $\delta \in \mathbb{C}$, $|\Lambda_{r,s}|$ is a lower bound of the dimension of the center of $B_{r,s}(\delta)$ (Proposition 3.5). Therefore, instead of solving the aforementioned system, which is very large and in most cases very difficult to be solved, one needs to find k linear independently equations in it.

In Proposition 2.5 we prove there is a bijection between the set $|\Lambda_{r,s}|$ and the set of some (r,s)-walled diagrams of $B_{r,s}(\delta)$. These diagrams have a particular generalized walled cycle type, which we define as *cycle type with trivial parts*. Using this result, we prove that the linear independently equations we need to find, could be obtained if for each generalized walled cycle type μ containing a non-trivial part, there is a diagram x with $c(x) = \mu$ (Proposition 3.6).

However, proving that these equations are linearly independent seems complicated for the moment for the general case. The main result of the paper is Theorem 4.8:

Theorem 1.2. *The dimension of the center of $B_{r,1}(\delta)$ is equal to $|\Lambda_{r,1}|$.*

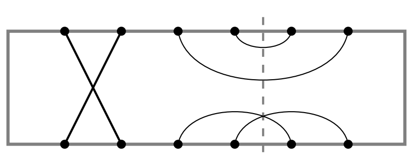
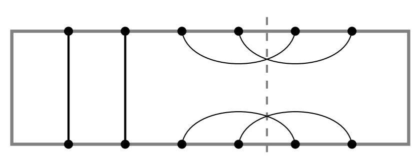
In the case of $B_{r,1}(\delta)$, we give a particular diagram x for each cycle type, which contains non-trivial parts. We associate to x two particular diagrams σ_x and z_x, whose walled generalized cycle types $c(\sigma_x)$ and $c(z_x)$ determine $c(x)$. These new diagrams are used in order to prove the linear independence.

2. Preliminaries

2.1. **Walled Brauer algebras.** Let r and s be nonnegative integers. An (r,s)-*walled Brauer diagram* is a graph drawn in a rectangle with $(r+s)$ vertices on its top and bottom edges, numbered $1, \ldots, r+s$ in order from left to right. Each vertex is connected by a strand to exactly one other vertex. In addition, there is a vertical wall separating the left r vertices form the right s vertices, such that the following conditions hold:

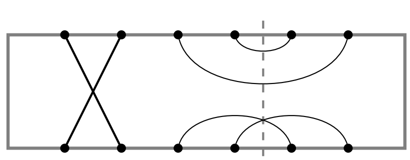
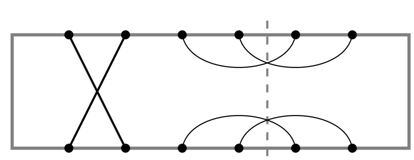
(1) A *propagating line* connects a vertex on the top row with one on the bottom row, and it cannot cross the wall.
(2) A *northern arc* (respectively, *southern arc*) connects vertices on the top row (respectively, on the bottom row), and it must cross the wall.

For example, the following graphs are $(4,2)$-Brauer diagrams:

$x :=$  , $y :=$ 

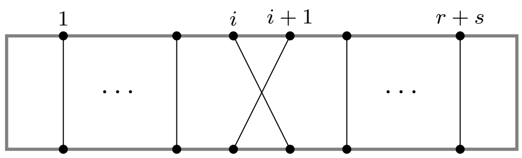
Let δ be a complex number. The *walled Brauer algebra* $B_{r,s}(\delta)$ is the $\mathbb{C}$-linear span of the (r,s)-walled diagrams with the multiplication defined as follows: The product of two (r,s)-walled Brauer diagrams d_1 and d_2 is determined by putting d_1 above d_2 and identifying the top vertices of d_1 with the bottom vertices of d_2. Let n be the number of loops in the middle row so obtained. The product $d_1 d_2$ is given by δ^n times the resulting diagram with loops omitted.

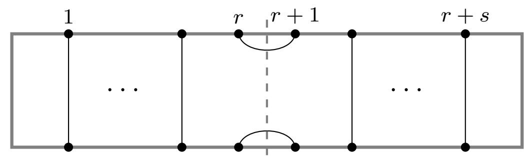
For example, for the diagrams x and y above we have:

$xy = \delta^2$  $yx = \delta$ 

The dimension of $B_{r,s}(\delta)$ equals $(r+s)!$ (see, for example, [2, 2.2]).

We denote by s_i $(1 \leq i \leq r-1$ or $r+1 \leq i \leq r+s-1)$ and e the following (r,s)-walled Brauer diagrams:

$s_i :=$ 

, $e :=$ 

Note that the $B_{0,n}(\delta) \simeq B_{n,0}(\delta) \simeq \mathbb{C}S_n$, the group algebra of the symmetric group S_n on n letters.

The algebra $B_{r,s}(\delta)$ is generated by the elements s_i $(1 \leq i \leq r-1$ or $r+1 \leq j \leq r+s-1)$ and e.

For any (r,s)-walled Brauer diagram d, we denote by d^* the *flip diagram* of d, obtained by horizontally flipping d. Therefore, we can define a $\mathbb{C}$-linear anti-automorphism $* : B_{r,s}(\delta) \to B_{r,s}(\delta)$.

The following theorem is Theorem 6.3 in [3] and it provides a criterion of the semisimplicity for the algebra $B_{r,s}(\delta)$.

Theorem 2.1. *The walled Brauer algebra $B_{r,s}(\delta)$ is semisimple if and only if one of the following holds:*

(1) $r = 0$ *or* $s = 0$*,*
(2) $\delta \notin \mathbb{Z}$*,*
(3) $|\delta| > r + s - 2$*,*
(4) $\delta = 0$*, and* $(r,s) \in \{(1,2), (1,3), (2,1), (3,1)\}$*.*

Therefore, for a fixed pair (r,s) the algebra $B_{r,s}(\delta)$ is semisimple except for finitely many values $\delta \in \mathbb{C}$.

2.2. **Walled generalized cycle types.** The following definition is Definition 7.3 in [11]. For a diagram $d \in B_{r,s}(\delta)$ we define the *walled generalized cycle type* $c(d)$ of d to be a set of words (called *parts*) in the alphabet L, R, N and S, obtained as follows: We first connect each vertex in the top row of d with the vertex in the bottom row below it. The parts of $c(d)$ correspond to the connected components of this new graph, as follows: We take a connected component of the new graph, we pick up a vertex of it and we follow the path, until all the edges of the connected component have been read off once. Following the path, we record in order with the letters L, R, N and S the types of the edges of the diagram d which are traversed. More precisely, we record:

- N if the type of the edge is a northern arc.
- S if the type of the edge is a southern arc.
- L if the type of the edge is a propagating line to the left of the wall.
- R if the type of the edge is a propagating line to the right of the wall.

For example, for the diagrams x and y we saw in Section 2.1 we have $c(x) = \{LL, NSNS\}$ and $c(y) = \{L, L, NS, NS\}$.

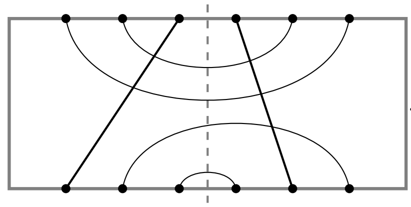
Two parts are *equivalent* if one is obtained from the other by repeated cyclic permutation and/or reverse reading. Two walled generalized cycle types are *equal* if their parts are equivalent. For example, we also have $c(y) = \{L, L, SN, SN\}$ and $c(x) = \{LL, SNSN\}$.

Remark 2.2. *From the definition of (equal) walled generalized cycle types we notice the following:*

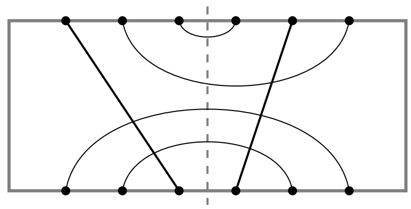
(1) Each part of $c(d)$ is of one of the following forms:
  (a) L^a, $a \geq 1$.
  (b) R^b, $b \geq 1$.
  (c) $NR^{b_1}SL^{a_1}NR^{b_2}SL^{a_2}\ldots,$ *for some* $a_i, b_i \geq 0$.

(2) The number of N's in $c(d)$ equals the number of S's in $c(d)$. We denote this number by t. Then, we also have that the number of L's in $c(d)$ equals $r - t$, while the number of R's in $c(d)$ equals $s - t$.

(3) *In general, $c(d) \neq c(d^*)$. For example, we consider the following diagram d of the walled Brauer algebra $B_{3,3}(\delta)$:*

$d =$  .

We have $c(d) = \{NSNRSL\}$. On the other hand,

$d^* =$ 

We have $c(d^*) = \{NSLNRS\}$. We notice that in $c(d^*)$ we have that NS is followed always by an R, while in $c(d)$ the NS is followed always by N. Hence, $c(d) \neq c(d^*)$.

The following theorem is Theorem 7.7 in [11].

Theorem 2.3. *Two diagrams of $B_{r,s}(\delta)$ are $S_r \times S_s$-conjugate if and only if they have equal walled generalized cycle types.*

Definition 2.4. *Let $d \in B_{r,s}(\delta)$ with cycle type $c(d)$. We call a part of $c(d)$ of type (a), (b) or (c) of the form NS in Remark 2.2(1) a* trivial part*.*

2.3. **Weights.** For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, set $|\lambda| := \sum_{i \geq 1} \lambda_i$. Let us denote Λ the set of bipartitions and let $k \in \mathbb{N}_0$. We set:

$$\Lambda_{r,s} := \coprod_{k=0}^{\min(r,s)} \{(\lambda, \mu) \in \Lambda \,|\, |\lambda| = r - k,\, |\mu| = s - k\},$$

$$\dot{\Lambda}_{r,s} := \begin{cases} \Lambda_{r,s} & \text{if } \delta \neq 0 \text{ or } r \neq s \text{ or } r = s = 0, \\ \Lambda_{r,s} - \{(\emptyset, \emptyset)\}, & \text{if } \delta = 0 \text{ and } r = s \neq 0. \end{cases}$$

The elements of $\dot{\Lambda}_{r,s}$ are called the *weights* of $B_{r,s}(\delta)$.

For each weight λ, there is an indecomposable module, $C_{r,s}(\lambda)$, called the *cell module*, which has an irreducible head $D_{r,s}(\lambda)$ and the family

$$\{D_{r,s}(\lambda) \,|\, \lambda \in \dot{\Lambda}_{r,s}\}$$

is the complete set of mutually non-isomorphic simple modules over $B_{r,s}(\delta)$ (see [3, Theorem 2.7]).

The next proposition associates weights with some particular diagrams of $B_{r,s}(\delta)$.

Proposition 2.5. *There is a bijection between the set $\Lambda_{r,s}$ and the set of diagrams in $B_{r,s}(\delta)$ having only trivial parts.*

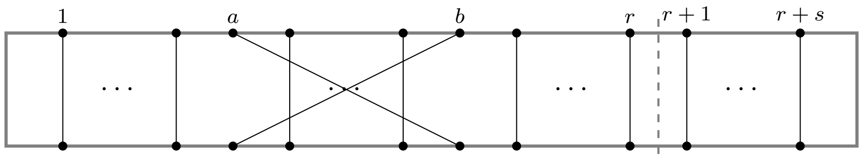
Proof. Let $k \in \mathbb{N}_0$ and let $\lambda = (\lambda_1, \lambda_2, \dots)$, $\mu = (\mu_1, \mu_2, \dots)$ partitions of $r-k$ and $s-k$, respectively. The following map is a bijection:

$$(\lambda, \mu) \mapsto \left\{ d \in B_{r,s}(\delta) \mid c(d) = \{L^{\lambda_1},\, L^{\lambda_2}, \dots, R^{\mu_1},\, R^{\mu_2}, \dots, \underbrace{NS,\, NS, \dots, NS}_{k \text{ terms}}\} \right\}.$$

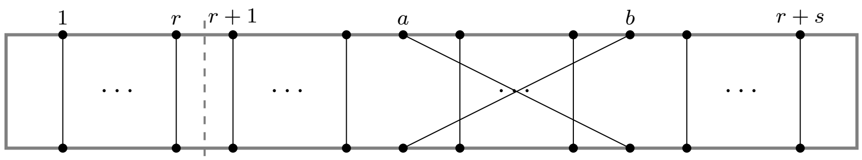
□

2.4. **Jucys-Murphy elements.** There are different definitions for Jucys-Murphy elements for the Brauer algebra $B_{r,s}(\delta)$. In this paper, we use Definition 2.1 of [5].

Let $(a, b) = (b, a)$ be the following diagram:

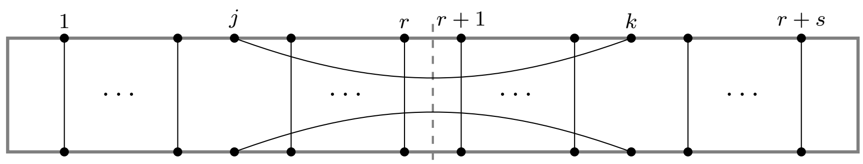


if $1 \le a < b \le r$,



if $r + 1 \le a < b \le r + s$,

Let also $e_{j,k}$, $(1 \le j \le r,\ r + 1 \le k \le r + s)$ be the diagram



For each $1 \le k \le r + s$ we define the *Jucys-Murphy elements* L_k of $B_{r,s}(\delta)$ as follows:

$$L_k := \begin{cases} 0 & \text{if } k = 1, \\ \sum_{j=1}^{k-1} (j, k) & \text{if } 1 < k \le r, \\ -\sum_{j=1}^{r} e_{j,k} + \sum_{j=r+1}^{k-1} (j, k) + \delta & \text{if } r + 1 \le k \le r + s. \end{cases}$$

We now collect some facts from Section 2 of [5], where we use the Jucys Murphy elements to describe the center of $B_{r,s}(\delta)$ for the semisimple case. In order to do that, we first need to recall the notion of *supersymmetric polynomials*.

Let m, n be nonnegative integers. An element p in the polynomial ring $\mathbb{C}[x_1, \dots, x_m, y_1, \dots, y_n]$ is sypersymmetric if

(1) p is symmetric in $x_1, \dots, x_m$ and $y_1, \dots, y_n$ separately.
(2) The substitution $x_m = t$, $y_1 = -t$ yields a polynomial in $x_1, \dots, x_{m-1}, y_2, \dots, y_m$ which is independent of t.

We denote by $\mathcal{S}_{m,n}[x; y]$ the set of supersymmetric polynomials in $x_1, \dots, x_m, y_1, \dots, y_n$.

The following result is Corollary 2.9 in [5] and it provides us with some central elements of $B_{r,s}(\delta)$.

Proposition 2.6. *For every supersymmetric polynomial p in $\mathcal{S}_{r,s}[x; y]$, the element*

$$p(L_1, \dots, L_r, L_{r+1}, \dots, L_{r+s})$$

belongs to the center of $B_{r,s}(\delta)$.

3. Facts on the center of $B_{r,s}(\delta)$

3.1. **The semisimple case.** We know that when an algebra is semisimple, the dimension of the center is the same as the number of the isomorphic classes of simple modules. Therefore, when $B_{r,s}(\delta)$ is semisimple, the dimension of the center is the cardinality of $\dot{\Lambda}_{r,s}$. Jung and Kim [5, Theorem 3.5] gave a basis of the center of $B_{r,s}(\delta)$. More precisely, they proved the following theorem.

Theorem 3.1. *If the walled Brauer algebra $B_{r,s}(\delta)$ is semisimple, the supersymmetric polynomials in $L_1, \ldots, L_{r+s}$ generate the center of $B_{r,s}(\delta)$. Moreover, there is a set of supersymmetric polynomials $\{p_\lambda \in S_{r,s}[x;y] \,|\, \lambda \in \dot{\Lambda}_{r,s}\}$, such that the set $\{p_\lambda(L_1, \ldots, L_{r+s}) \,|\, \lambda \in \dot{\Lambda}_{r,s}\}$ is a basis of the center of $B_{r,s}(\delta)$.*

One can find a precise description of these supersymmmetric polynomials p_λ in [5, Section 3].

The goal of this paper is to describe the center of $B_{r,s}(\delta)$ in the non-semisimple case. More precisely, we want to prove the following conjecture [5, Conjecture 5.4]:

Conjecture 3.2. *For every $\delta \in \mathbb{C}$, the center of the walled Brauer algebra $B_{r,s}(\delta)$ is generated by the supersymmetric polynomials in the Jucys-Murphy elements $L_1, \ldots, L_n$.*

3.2. **Towards general case.** Let z be a formal parameter. We consider the walled Brauer algebra $B_{r,s}^{\mathbb{C}[z]}(z)$ over the polynomial ring $\mathbb{C}[z]$.

We now define the Jucys-Murphy elements $\mathcal{L}_i$, $1 \leq i \leq r+s$ as in Section 2.4, replacing δ by z. One can repeat the proof of Proposition 2.6 and show the following:

Proposition 3.3. *All supersymmetric polynomials (with coefficients in $\mathbb{C}[z]$) in $\mathcal{L}_1, \ldots, \mathcal{L}_{r+s}$ are central in $B_{r,s}^{\mathbb{C}[z]}(z)$.*

We now take a $\delta_0 \in \mathbb{C}$ ($\delta_0 \neq 0$), such that $B_{r,s}(\delta_0)$ is semisimple (see Theorem 2.1). We set $m := |\Lambda_{r,s}|$ and we choose $P_i \in S_{r,s}[x;y]$ for $1 \leq i \leq m$ to be supersymmetric polynomials such that

$$\{P_i(L_1, \ldots, L_{r+s}) \,|\, 1 \leq i \leq m\}$$

form a $\mathbb{C}$-basis for the center $Z(B_{r,s}(\delta_0))$ of $B_{r,s}(\delta_0)$.

Let $\mathbb{C}(z)$ be the field of fractions of $\mathbb{C}[z]$ and we consider the algebra

$$B_{r,s}^{\mathbb{C}(z)}(z) := B_{r,s}^{\mathbb{C}[z]}(z) \otimes_{\mathbb{C}[z]} \mathbb{C}(z).$$

Using the same arguments of Theorem 6.3 in [3], we have that $B_{r,s}^{\mathbb{C}(z)}(z)$ is a semisimple $\mathbb{C}(z)$-algebra with m simple modules, hence

$$\dim_{\mathbb{C}(z)} Z(B_{r,s}^{\mathbb{C}(z)}(z)) = m.$$

Moreover,

$$Z(B_{r,s}^{\mathbb{C}(z)}(z)) = Z(B_{r,s}^{\mathbb{C}[z]}(z)) \otimes_{\mathbb{C}[z]} \mathbb{C}(z).$$

Therefore,

$$\operatorname{rank}_{\mathbb{C}[z]} Z(B_{r,s}^{\mathbb{C}[z]}(z)) = \dim_{\mathbb{C}(z)} Z(B_{r,s}^{\mathbb{C}(z)}(z)) = m.$$

Proposition 3.4. *The set $\{P_i(\mathcal{L}_1, \ldots, \mathcal{L}_{r+s}) \,|\, 1 \leq i \leq m\}$ forms a $\mathbb{C}[z]$-basis for $Z(B_{r,s}^{\mathbb{C}[z]}(z))$.*

Proof. From Proposition 3.3 we have that $P_i(\mathcal{L}_1, \ldots, \mathcal{L}_{r+s}) \in Z(B_{r,s}^{\mathbb{C}[z]}(z))$, for every $i = 1, \ldots, m$. Moreover, $\operatorname{rank}_{\mathbb{C}[z]} Z(B_{r,s}^{\mathbb{C}[z]}(z)) = m$. Therefore, it is enough to show that this set is linearly independent.

Assume $\sum_{i=1}^{m} a_i(z) P_i(\mathcal{L}_1, \ldots, \mathcal{L}_{r+s}) = 0$, for some $a_i(z) \in \mathbb{C}[z]$, not all 0. Dividing by the highest power of $(z - \delta_0)$ which divides all the $a_i(z)$'s, we can assume that there exists some i_0 with $a_{i_0}(\delta_0) \neq 0$.

Now, specializing to $z = \delta_0$, and noting that under this specialization the elements $\mathcal{L}_i$'s become the L_i's, we get

$$\sum_{i=1}^{m} a_i(\delta_0) P_i(L_1, \ldots, L_{r+s}) = 0.$$

Now, as $\{P_i(L_1,\ldots,L_{r+s})\,|\,1\le i\le m\}$ is a linearly independent set, we must have $a_i(\delta_0)=0$, $\forall i=1,\ldots,m$, which contradicts the fact that $a_{i_0}(\delta_0)\neq 0$. ☐

We now take an arbitrary $\delta\in\mathbb{C}$ and we consider

$$B^{\mathbb{C}}_{r,s}(\delta)=B^{\mathbb{C}[z]}_{r,s}(z)\otimes_{\mathbb{C}[z]}\mathbb{C}[z]/\langle z-\delta\rangle.$$

For any $Q\in B^{\mathbb{C}[z]}_{r,s}(z)$, we write

$$\bar{Q}:=Q\otimes_{\mathbb{C}[z]}\mathbb{C}[z]/\langle z-\delta\rangle\in B^{\mathbb{C}}_{r,s}(\delta).$$

To simplify notation we write

$$\mathcal{P}_i:=P_i(\mathcal{L}_1,\ldots,\mathcal{L}_{r+s})\in B^{\mathbb{C}[z]}_{r,s}(z),\ i=1,\ldots,m.$$

Proposition 3.5. *For any choice of $\delta\in\mathbb{C}$, the set $\{\bar{\mathcal{P}}_i\,|\,1\le i\le m\}$ is a linearly independent set in $Z(B^{\mathbb{C}}_{r,s}(z))$. In particular,* $\dim_{\mathbb{C}} Z(B^{\mathbb{C}}_{r,s}(z))\ge|\Lambda_{r,s}|$.

Proof. First note that as $\mathcal{P}_i$ are central elements in $B^{\mathbb{C}[z]}_{r,s}(z)$ we have that $\bar{\mathcal{P}}_i$ are central in $B^{\mathbb{C}}_{r,s}(z)$. Now let

$$\sum_{i=1}^m a_i\bar{\mathcal{P}}_i=0$$

for some $a_i\in\mathbb{C}$. Then, we have

$$\sum_{i=1}^m a_i\bar{\mathcal{P}}_i=\sum_{i=1}^m\overline{a_i\mathcal{P}_i}=0$$

and so

$$\sum_{i=1}^m a_i\mathcal{P}_i=(z-\delta)R \tag{3.1}$$

for some $R\in Z(B^{\mathbb{C}[z]}_{r,s}(z))$. Thus, R can be written as $R=\sum_{i=1}^m b_i(z)\mathcal{P}_i$, for some $b_i(z)\in\mathbb{C}[z]$. Using now Equation (3.1) we get $\sum_{i=1}^m a_i\mathcal{P}_i=\sum_{i=1}^m(z-\delta)b_i(z)\mathcal{P}_i$. As $a_i\in\mathbb{C}$, $b_i(z)\in\mathbb{C}[z]$ and $\{\mathcal{P}_i\,|\,1\le i\le m\}$ are linearly independent, we must have $a_i=(z-\delta)b_i(z)=0$, $\forall i=1,\ldots,m$. ☐

Let $\mathcal{B}$ be the diagram basis of $B_{r,s}(\delta)$. For each walled generalized cycle type μ we set $C_\mu:=\{d\in\mathcal{B}\,|\,c(d)=\mu\}$.

In [11, Section 7] there is a description of a basis of the centralizer of $Z_{B_{r,s}(\delta)}(\mathbb{C}(\mathfrak{S}_r\times\mathfrak{S}_s))$. More precisely, each element of the basis corresponds to a walled generalized cycle type μ and it is of the form $\sum_{d\in C_\mu} d$.

Recall from Section 2.1 that the walled Brauer algebra $B_{r,s}(\delta)$ is generated by the generators of $\mathfrak{S}_r\times\mathfrak{S}_s$ and by the generator e. A central element of $B_{r,s}(\delta)$ is therefore an element of $Z_{B_{r,s}(\delta)}(\mathbb{C}(\mathfrak{S}_r\times\mathfrak{S}_s))$ which commutes with the generator e. Let us consider an element of $Z_{B_{r,s}(\delta)}(\mathbb{C}(\mathfrak{S}_r\times\mathfrak{S}_s))$, which is of the form $\sum_\mu a_\mu\sum_{d\in C_\mu}d$, for some $a_\mu\in\mathbb{C}$. Therefore, in order to find the center of $B_{r,s}(\delta)$, we consider the equation

$$\sum_\mu a_\mu\sum_{d\in C_\mu}(de-ed)=\sum_{x\in\mathcal{B}}b_x x. \tag{3.2}$$

Our goal is to prove that the rank of the system

$$\{b_x=0\,|\ \text{for all } x\in\mathcal{B}\}. \tag{3.3}$$

is $|\Lambda_{r,s}|$. The next proposition gives us a condition for this to be true.

Proposition 3.6. *If for each cycle type μ containing a non-trivial part, there is a diagram x with $c(x) = \mu$ such that $\{b_x = 0, \mid$ for all such $x\}$ are linearly independent, then the system (3.3) is of rank $|\Lambda_{r,s}|$.*

Proof. Since the coefficients b_x are linear combinations of the coefficients a_μ, the system (3.3) can be viewed as a set of linear equations in the variables a_μ. Therefore, the dimension of the center equals

$$\dim Z(B_{r,s}(\delta)) = \#\text{ cycle types for } B_{r,s} - \text{rank of the system (3.3)}$$

Our goal is to prove that $\dim Z(B_{r,s}(\delta)) = |\Lambda_{r,s}|$. From Proposition 3.5 it is enough to prove that

$$\#\text{ cycle types for } B_{r,s} - |\Lambda_{r,s}| \leq \text{rank of the system (3.3)},$$

since this would imply that

$$\dim Z(B_{r,s}(\delta)) \leq \#\text{ cycle types for } B_{r,s} - (\#\text{ cycle types for } B_{r,s} - |\Lambda_{r,s}|) = |\Lambda_{r,s}|,$$

as required.

In order to prove that, we need to find # cycle types for $B_{r,s} - |\Lambda_{r,s}|$ linearly independent equations in (3.3). The result now follows from our assumption and Proposition 2.5. ☐

4. The center of $B_{r,1}(\delta)$

Let $\mathcal{B}$ be the diagram basis of $B_{r,1}(\delta)$, consisting of $(r+1)!$ diagrams. We recall that for each walled generalized cycle type μ we set $C_\mu = \{d \in \mathcal{B} \mid c(d) = \mu\}$.

The following lemma is easy to see and it is true only for the case of $B_{r,1}(\delta)$ (see Remark 2.2 (3)).

Lemma 4.1. *The flip map $*$ gives a bijection $C_\mu \to C_\mu$, $d \mapsto d^*$ for each μ.*

We now consider equation (3.2) and system (3.3). Our first attempt to solve this system is to find diagrams $x \in B_{r,1}(\delta)$ for which we have $b_x = 0$ (and, hence, simplify the undermentioned system).

The first result on this direction is the following proposition:

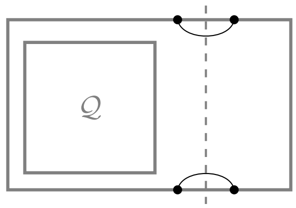
Proposition 4.2. *$b_{x^*} = -b_x$. Therefore, if $x = x^*$ then $b_x = 0$.*

Proof. We apply the anti-automorphism $*$ to both sides of (3.2) and we have:

$$\begin{aligned}
\sum_\mu a_\mu \sum_{d \in C_\mu} (de - ed)^* &= \sum_{x \in \mathcal{B}} b_x x^* \implies \\
\sum_\mu a_\mu \sum_{d \in C_\mu} (e^* d^* - d^* e^*) &= \sum_{x \in \mathcal{B}} b_x x^* \overset{e^*=e}{\implies} \\
\sum_\mu a_\mu \sum_{d \in C_\mu} (ed^* - d^* e) &= \sum_{x \in \mathcal{B}} b_x x^* \overset{4.1}{\implies} \\
\sum_\mu a_\mu \sum_{d \in C_\mu} (ed - de) &= \sum_{x \in \mathcal{B}} b_x x^* \implies \\
-\sum_\mu a_\mu \sum_{d \in C_\mu} (de - ed) &= \sum_{x \in \mathcal{B}} b_x x^* \implies \\
-\sum_{x \in \mathcal{B}} b_x x &= \sum_{x \in \mathcal{B}} b_x x^* \implies \\
-\sum_{x \in \mathcal{B}} b_x x &= \sum_{x \in \mathcal{B}} b_{x^*} x.
\end{aligned}$$

☐

The next proposition is another case of a basis diagram x, such that $b_x = 0$.

Proposition 4.3. *Let* $x =$ 

$\in B_{r,1}(\delta)$, *then* $b_x = 0$.

Proof. Let $d \in \mathcal{B}$ such that $de = \delta^m x$, $m \in \{0, 1\}$. We have the following cases:

(1) $d = \mathcal{Q} \in S_{r-1}$, with $\mathcal{Q}$ as defined in the diagram of x. We have $de = x$.
(2) $d = x$. We have $de = \delta x$.
(3) $d = x(i, r)$, where (i, r) corresponds to the permutation diagram swapping i and r. We have $de = x$.

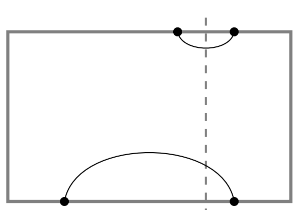
We notice that for every diagram d as described above, there is a diagram $d' \in \mathcal{B}$ such that $ed' = \delta^m x$, $m \in \{0, 1\}$. More precisely, for the cases (1) and (2), the diagram $d' = d$. For case (3), we have $d' = (i, r)x$. We notice that the diagrams d' we have here are all the diagrams $d' \in \mathcal{B}$ such that $de = \delta^m x$, $m \in \{0, 1\}$. In order to prove that $b_x = 0$ it is enough to prove that in each case, d and d' have the same walled generalized cycle type. Then the terms in the sum in equation (3.2) cancel out and, hence, $b_x = 0$. For cases (1) and (2) this is obvious. It remains to prove that the diagrams $x(i, r)$ and $(i, r)x$ have the same walled generalized cycle type.

There are two types of parts in the walled generalized cycle type $c(x)$ of the diagram x: The part $p_1 = NS$ and the parts p_j, $j = 2, 3, \ldots, k$, which are of one of the forms: L, LL, LLL, $\ldots$. Let p_{j_0} be the part, which belongs to the connected component with vertex i on top.

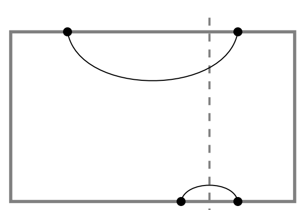
In the diagram of d there is a northern arc, which connects the vertices r and $r + 1$ on top row and a southern arc, which connects the vertices i and r on bottom row. The propagating lines are the same as the ones appearing in the diagram of x, with one difference: The propagating line which connects the vertex i on bottom row with the vertex i' on top row in the diagram of x, it connects now the vertex r on bottom row with the vertex i' on top row. Therefore, $c(d) = \{NSp_{j_0}, p_2,\, p_3,\, p_{j_0-1},\, p_{j_0+1}, \ldots, p_k\}$.

Similarly, in the diagram of d' there are two arcs, which are the flipped arcs of the diagram of d and the propagating lines remaining from the diagram of x with again one difference: the propagating line which connects the vertex i on top row with the vertex i' on bottom row in the diagram of x, it connects now the vertex r on top row with the vertex i' on the bottom row. Therefore, we have $c(d') = \{NSp_{j_0}, p_2,\, p_3,\, p_{j_0-1},\, p_{j_0+1}, \ldots, p_k\} = c(d)$. ☐

We consider now the system (3.3). According to Proposition 4.3, the only possible diagrams x with $b_x \neq 0$ are of the form

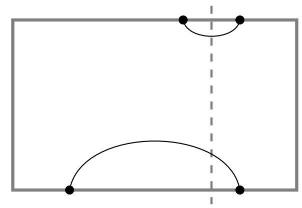


or



Note that the second case can be obtained from the first by applying the flip map $*$. Therefore, by Proposition 4.2, it is enough to consider only the first case.

Proposition 4.4. *Let $x, y \in B_{r,1}(\delta)$ be of the form*



with $y = \sigma x \sigma^{-1}$, for some $\sigma \in S_r$. Then, $b_x = b_y$.

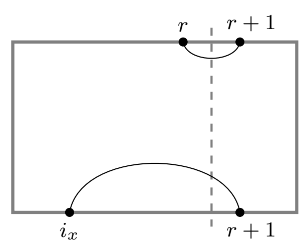
Proof. We first notice that, since both diagrams x and y have the northern arc that connects the vertices r and $r+1$, we must have $\sigma \in S_{r-1}$. Therefore, $\sigma e = e\sigma$ and $\sigma^{-1}e = e\sigma^{-1}$.

We now conjugate Equation (4.1) by σ and we get:

$$\begin{aligned}
\sum_{\mu} a_\mu \sum_{d \in C_\mu} \sigma^{-1}(de - ed)\sigma &= \sum_{x \in \mathcal{B}} b_x \sigma^{-1} x \sigma \implies \\
\sum_{\mu} a_\mu \sum_{d \in C_\mu} (\sigma^{-1} de\sigma - \sigma^{-1} ed\sigma) &= \sum_{x \in \mathcal{B}} b_x \sigma^{-1} x \sigma \implies \\
\sum_{\mu} a_\mu \sum_{d \in C_\mu} (\sigma^{-1} d\sigma e - e\sigma^{-1} d\sigma) &= \sum_{x \in \mathcal{B}} b_x \sigma^{-1} x \sigma \overset{2.3}{\implies} \\
\sum_{\mu} a_\mu \sum_{d \in C_\mu} (de - ed) &= \sum_{x \in \mathcal{B}} b_x \sigma^{-1} x \sigma \implies \\
\sum_{x \in \mathcal{B}} b_x x &= \sum_{x \in \mathcal{B}} b_{\sigma x \sigma^{-1}} x.
\end{aligned}$$

□

Definition 4.5. *Let $x \in B_{r,1}(\delta)$ be of the form*



where $i_x \neq r$. The diagram x defines the following bijection, given by the propagating lines in the diagram x:

$$f_x : \{1, \dots, r-1\} \to \{1, \dots, \widehat{i_x}, \dots, r\}.$$

We associate two diagrams in $B_{r,1}(\delta)$ to x as follows:

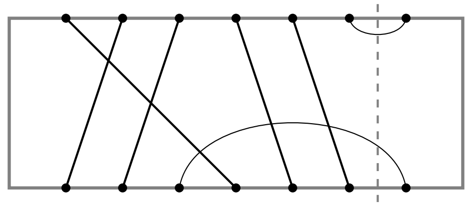
(i) Define $\sigma_x \in S_r \times S_1 \subseteq B_{r,1}(\delta)$ by

$$\sigma_x(k) = \begin{cases} f_x(k), & \text{if } k \in \{1, \dots, r-1\} \\ i_x, & \text{if } k = r \\ r+1, & \text{if } k = r+1 \end{cases}$$

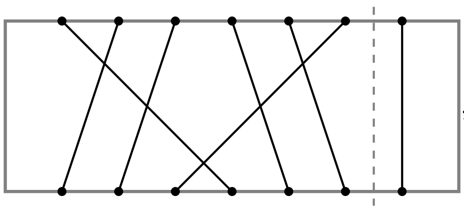
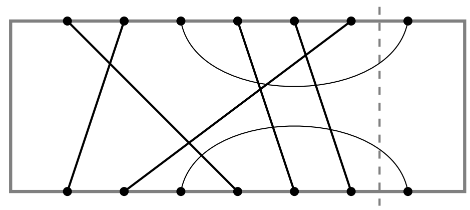
(ii) Define $z_x \in B_{r,1}(\delta)$ as follows: The vertices i_x and $r+1$ on top row (respectively, on bottom row) are connected by a northern arc (respectively, a southern arc). The propagating lines are given by the permutation $\tau_x : \{1, \dots, \widehat{i_x}, \dots, r\} \to \{1, \dots, \widehat{i_x}, \dots, r\}$, defined by

$$\tau_x(k) = \begin{cases} f_x(k), & \text{if } k \neq r \\ f_x(i_x), & \text{if } k = r \end{cases}$$

For example, let $x \in B_{6,1}(\delta)$ be the diagram



Then, we have: $f_x(1) = 4$, $f_x(2) = 1$, $f_x(3) = 2$, $f_x(4) = 5$, $f_x(5) = 6$. Therefore:

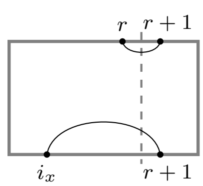
$\sigma_x =$  , $\quad z_x =$ 

Proposition 4.6. *Let x, σ_x and z_x be as in 4.5, then we have:*

(i) $e\sigma_x = x$. Moreover, $c(\sigma_x)$ is obtained from $c(x)$ by replacing NS by L and σ_x is the unique diagram $y \in B_{r,1}(\delta)$ with $c(y)$ without NS satisfying $ey = x$.
(ii) $ez_x = x$. Moreover, $c(z_x)$ is obtained from $c(x)$ by removing NS and adding it back as a trivial part and z_x is the unique diagram $y \in B_{r,1}(\delta)$ with $c(y)$ having NS only as trivial part satisfying $ey = x$.

Proof. The result follows by definition of σ_x, z_x, concatenation of diagrams and the definition of walled generalized cycle type. $\square$

Corollary 4.7. *The set of equations*

$\Bigg\{ b_x = 0 \,|\, x =$  $i_x \neq r,$ *one x for each walled generalized cycle type* $\Bigg\}$

viewed as equations in variables a_μ as defined in equation (3.2) are linearly independent.

Proof. We represent the aforementioned equations by a matrix, whose rows are the b_x's and columns the coefficients a_μ.

By Proposition 4.6, each b_x has a unique factor a_μ appearing with coefficient 1, such that μ has no NS, namely $a_{c(\sigma_x)}$, and a unique factor $a_{\mu'}$ (appearing also with coefficient 1), such that μ' has NS as a trivial part, namely $a_{c(z_x)}$.

Now, note that $c(\sigma_x)$ and $c(z_x)$ determine $c(x)$, therefore the matrix cannot have the following form:

$$\left[\begin{array}{cccccccccccc} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & * & * & * & * \\ 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & * & * & * & * \\ & & \vdots & & & & \vdots & & & \vdots & & \end{array}\right]$$
$$\underbrace{\hphantom{1\ 0\ \cdots\ 0}}_{\text{no NS}} \qquad \underbrace{\hphantom{1\ 0\ \cdots\ 0}}_{\text{trivial NS}} \qquad \underbrace{\hphantom{*\ *\ *\ *}}_{\text{non-trivial NS}}$$

Therefore, the matrix is of the following form and, hence, the equations linearly independent.

$$
\left[\begin{array}{cccccccccccc}
1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & * & * & * & * \\
1 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & * & * & * & * \\
0 & 1 & \cdots & 0 & 1 & 0 & \cdots & 0 & * & * & * & * \\
0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 & * & * & * & * \\
 & & \vdots & & & & \vdots & & & \vdots & &
\end{array}\right]
$$
$$
\underbrace{\qquad\qquad}_{\text{no NS}} \quad \underbrace{\qquad\qquad}_{\text{trivial NS}} \quad \underbrace{\qquad\qquad}_{\text{non-trivial NS}}
$$

□

Theorem 4.8. *The dimension of the center of $B_{r,1}(\delta)$ is equal to $|\Lambda_{r,1}|$.*

Proof. The result follows from Proposition 3.6 and Corollary 4.7. □

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Institut für Diskrete Strukturen und Symbolisches Rechnen, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany.
Email address: `eirini.chavli@mathematik.uni-stuttgart.de`

Department of Mathematics, City University of London, Northampton Square, London EC1V 0HB, United Kingdom.
Email address: `maud.devisscher.1@city.ac.uk`

School of Mathematics, University of Leeds, LS2 9JT, United Kingdom.
Email address: `a.e.parker@leeds.ac.uk`

Department of Mathematics, University of Colorado Boulder, Campus Box 395, 80309-0395, United States.
Email address: `Sarah.Salmon@colorado.edu`

Science, Technology, Engineering and Mathematics Division Faculty, Morehouse College,830 Westview Drive, S.W. Atlanta, GA 30314, United States.
Email address: `ulrica.wilson@morehouse.edu`