Deformations of the finite quotients of the braid group on 3 strands

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Complex Reflection Groups, Braid Groups, Hecke algebras

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- Any finite Coxeter group,

$$C =< s_1, ..., s_n \mid s_i^2 = 1, (s_i s_j)^{m_{i,j}} = 1 >$$

  Then, $C \leq GL_n(\mathbb{R})$, as a reflection group.

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- J is the two-sided ideal generated by the $(\sigma - u_{s,1}) \ldots (\sigma - u_{s,n})$, where σ runs through all braided reflections associated to s.

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$$H =< s_1, s_2 \mid s_1s_2s_1 = s_2s_1s_2,\ (s_i - a)(s_i - b) = 0 >$$

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The Hecke algebra of every irreducible complex reflection group is generated as R-module by $|W|$ elements.

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Among them there are 22 of rank 2.

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Weak version of the conjecture

The conjecture is true over the field of fractions of R.

The exceptional groups: The case of the finite quotients of B_n

Theorem (Coxeter 1957)
The quotient W of the B_n by the relations $s_i^k = 1$ is a finite group if and only if $\frac{1}{k} + \frac{1}{n} > \frac{1}{2}$.

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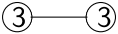
- $n = 3$ and $k = 3, 4, 5$. Then $W = G_4,\ G_8,\ G_{16}$.

The cases of G_4, G_8 and G_{16}

$$G_4 = B_3 / < s_i^3 >$$



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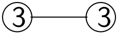

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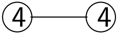

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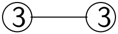

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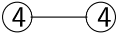

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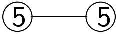

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  Therefore, $H_4 = U$, which is generated as u_1-module by 8 elements. H_4 is generated as R_4-module by $|G_4| = 24$ elements.
- The case of H_8

$$H_8 = u_1 + u_1\omega + u_1\omega^{-1} + u_1\omega^{-2} + u_1 u_2 u_1 + u_1 s_2^{-1} s_1 s_2^{-1} u_1 + u_1 s_2 s_1^{-1} s_2 u_1$$

The cases of G_4, G_8 and G_{16}

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- The case of G_{16}

$$
\begin{aligned}
H_{16} = \; & u_1 + u_1\omega + u_1\omega^{-1} + u_1\omega^2 + u_1\omega^{-2} + u_1\omega^3 + u_1\omega^{-3} + \\
& + u_1\omega^4 + u_1\omega^{-4} + u_1\omega^{-5} + u_1 u_2 u_1 + u_1 s_2^{-1} s_1^2 s_2^{-1} u_1 + \\
& + u_1 s_2 s_1^{-2} s_2 u_1 + u_1 s_2^2 s_1^2 s_2^2 u_1 + u_1 s_2^{-2} s_1^{-2} s_2^{-2} u_1 + \\
& + u_1 s_2 s_1^{-2} s_2^2 u_1 + u_1 s_2^{-1} s_1^2 s_2^{-2} u_1 + u_1 s_2^{-1} s_1 s_2^{-1} u_1 + \\
& + u_1 s_2 s_1^{-1} s_2 u_1 + u_1 s_2^{-2} s_1^{-2} s_2^2 u_1 + u_1 s_2^2 s_1^2 s_2^{-2} u_1 + \\
& + u_1 s_2^2 s_1^{-2} s_2^2 u_1 + u_1 s_2^{-2} s_1^2 s_2^{-2} u_1 + u_1 s_2^{-2} s_1 s_2^{-1} u_1 + \\
& + u_1 s_2^{-1} s_1 s_2^{-2} u_1 + u_1 s_2^{-2} s_1^2 s_2^{-1} s_1 s_2^{-1} u_1 + \\
& + u_1 s_2^2 s_1^{-2} s_2 s_1^{-1} s_2 u_1 + u_1 s_2 s_1^{-2} s_2^2 s_1^{-2} s_2^2 u_1 + \\
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\end{aligned}
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Lemma
For every $\varepsilon \in \mathbb{Z}^+$ we have

$$\begin{aligned} s_2 s_1^{\varepsilon} s_2^{-1} &= s_1^{-1} s_2^{\varepsilon} s_1 \\ s_2 s_1^{-\varepsilon} s_2^{-1} &= s_1^{-1} s_2^{-\varepsilon} s_1 \\ s_2^{-1} s_1^{\varepsilon} s_2 &= s_1 s_2^{\varepsilon} s_1^{-1} \\ s_2^{-1} s_1^{-\varepsilon} s_2 &= s_1 s_2^{-\varepsilon} s_1^{-1} \end{aligned}$$

```
gap> W := ComplexReflectionGroup(8);;
gap> W0 := ReflectionSubgroup(W,[1]);;
gap> Size(W0);
4
gap> dc := DoubleCosets(W,W0,W0);;
gap> Length(dc);
9
gap> dce := List(dc,Elements);;
gap> First([1..9],y->() in dce[y]);
1
gap> First([1..9],y->W.2 in dce[y]);
6
gap> First([1..9],y->W.2^(-1) in dce[y]);
5
gap> First([1..9],y->W.2^(-2) in dce[y]);
9
gap> First([1..9],y->W.2^(-1)*W.1*W.2^(-1) in dce[y]);
8
gap> First([1..9],y->W.2*W.1^(-1)*W.2 in dce[y]);
```

```
7
gap> First([1..9],y->W.2*W.1^(-2)*W.2 in dce[y]);
3
gap> List(dce,Length);
[ 4, 4, 4, 4, 16, 16, 16, 16, 16 ]
gap> zzw := Centre(W);;
gap> Size(zzw);
4
gap> ezzw := Elements(zzw);;
gap> List(ezzw,u->First([1..9],y->u in dce[y]));
[ 1, 3, 4, 2 ]
```

Symmetric algebras

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Example:
In the exemple where $A = \mathbb{Z}[G]$, $s_\chi = \frac{|G|}{\chi(1)}$.

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Let $\mathcal{O} \subset K$ be a valuation ring with $R \subset \mathcal{O}$.
We assume that L is the residue field of $\mathcal{O}$.
For every KA-module V there is an $\mathcal{O}A$-lattice $\tilde{V}$ such that $K\tilde{V} := \tilde{V} \otimes_R K = V$.

Definition The additive map $d_\theta : R_0^+(KA) \to R_0^+(LA)$, $[V] \mapsto [L\tilde{V}]$, is called the decomposition map associated with the specialisation θ.

The representations of B_3 of dimension ≤ 5

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Let $A := \rho(s_1)$ and $B := \rho(s_2)$.
By the theorem of Cayley-Hamilton there is a polynomial $Q(X) \in k[X]$ of degree d such that $Q(A) = Q(B) = 0$.

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We only need to describe the $kH_d := H_d \otimes_\theta k$-modules.

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There is an irreducible K_dH_d module M such that $d_\theta([M]) = a[S] + [J]$, where $a > 0$ and J is an L_dH_d module.

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- $d = 2$. Then, $d_\theta([M]) = [S]$, where M is the 2-dimensional $K_2 H_2$ module.

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The irreducible representations of B_3 of dimension $d = 2, 3, 4, 5$

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The representations of B_3 of dimension ≤ 5

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Theorem (Geck 1990)
If $\theta(s_{\chi_V}) \neq 0$ for $\chi_V \in Irr(KA)$, then $d_\theta([V])$ is the class of a simple module in $R_0^+(LA)$.

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The inverse is not always true.

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Let V be any 1-dimensional $K_3 H_3$ module. Then $d_\theta([V])$ is the class of a simple module in $R_0^+(L_3 H_3)$, but $\theta(s_{\chi_V}) = 0$.

- $d = 5$. We suppose that $\theta(s_\chi) \neq 0$, for every 6-dimensional module of $K_5 H_5$.Then, $d_\theta([M]) = [S]$, where M is one of the 5-dimensional $K_5 H_5$ modules.

Remarque (Tuba, Wenzl 1999) For the above modules the inverse of the theorem of Geck is true.

Last slide

Thank You!