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Universal deformations of the finite quotients of the braid group on 3 strands

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ABSTRACT

We prove that the quotients of the group algebra of the braid group on 3 strands by a generic quartic and quintic relation respectively have finite rank. This is a special case of a conjecture by Broué, Malle and Rouquier for the generic Hecke algebra of an arbitrary complex reflection group. Exploring the consequences of this case, we prove that we can determine completely the irreducible representations of this braid group of dimension at most 5, thus recovering a classification of Tuba and Wenzl in a more general framework.

1. Introduction

In 1999 I. Tuba and H. Wenzl classified the irreducible representations of the braid group B_3 of dimension k at most 5 over an algebraically closed field K of any characteristic (see [19]) and, therefore, of $PSL_2(\mathbb{Z})$, since the quotient group B_3 modulo its center is isomorphic to $PSL_2(\mathbb{Z})$. Recalling that B_3 is given by generators s_1 and s_2 that satisfy the relation $s_1s_2s_1 = s_2s_1s_2$, we assume that $s_1 \mapsto A$, $s_2 \mapsto B$ is an irreducible representation of B_3, where A and B are invertible $k \times k$ matrices over K satisfying

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$ABA = BAB$. I. Tuba and H. Wenzl proved that A and B can be chosen to be in *ordered triangular form*[1] with coefficients completely determined by the eigenvalues (for $k \leq 3$) or by the eigenvalues and by the choice of a kth root of $\det A$ (for $k > 3$). Moreover, they proved that such irreducible representations exist if and only if the eigenvalues do not annihilate some polynomials P_k in the eigenvalues and the choice of the kth root of $\det A$, which they determined explicitly.

At this point, a number of questions arise: what is the reason we do not expect their methods to work for any dimension beyond 5 (see [19], Remark 2.11, 3)? Why are the matrices in this neat form? In [19], Remark 2.11, 4 there is an explanation for the nature of the polynomials P_k. However, there is no argument connected with the nature of P_k that explains the reason why these polynomials provide a necessary condition for a representation of this form to be irreducible. In this paper we answer these questions by recovering this classification of the irreducible representations of the braid group B_3 as a consequence of the freeness conjecture for the generic Hecke algebra of the finite quotients of the braid group B_3, defined by the additional relation $s_i^k = 1$, for $i = 1, 2$ and $2 \leq k \leq 5$. For this purpose, we first prove this conjecture for $k = 4, 5$ (the rest of the cases are known by previous work). The fact that there is a connexion between the classification of the irreducible representations of B_3 of dimension at most 5 and its finite quotients has already been suspected by I. Tuba and H. Wenzl (see [19], Remark 2.11, 5).

More precisely, there is a Coxeter's classification of the finite quotients of the braid group B_n on n strands by the additional relation $s_i^k = 1$, for $i = 1, 2$ (see [7]); these quotients are finite if and only if $\frac{1}{k} + \frac{1}{n} > \frac{1}{2}$. If we exclude the obvious cases $n = 2$ and $k = 2$, which lead to the cyclic groups and to the symmetric groups respectively, there is only a finite number of such groups, which are irreducible complex reflection groups: these are the groups G_4, G_8 and G_{16}, for $n = 3$ and $k = 3, 4, 5$ and the groups G_{25}, G_{32} for $n = 4, 5$ and $k = 3$, as they are known in the Shephard–Todd classification (see [18]). Therefore, if we restrict ourselves to the case of B_3, we have the finite quotients W_k, for $2 \leq k \leq 5$, which are the groups $\mathfrak{S}_3$, G_4, G_8 and G_{16}, respectively.

We set $R_k = \mathbb{Z}[a_{k-1}, ..., a_1, a_0, a_0^{-1}]$, for $k = 2, 3, 4, 5$ and we denote by H_k the *generic Hecke algebra* of W_k; that is the quotient of the group algebra $R_k B_3$ by the relations $s_i^k = a_{k-1} s_i^{k-1} + ... + a_1 s_i + a_0$, for $i = 1, 2$. We assume we have an irreducible representation of B_3 of dimension k at most 5. By the Cayley–Hamilton theorem of linear algebra, the image of a generator under such a representation is annihilated by a monic polynomial $m(X)$ of degree k, therefore this representation has to factorize through the corresponding Hecke algebra H_k. As a result, if $\theta : R_k \to K$ is a specialization of H_k such that $a_i \mapsto m_i$, where m_i are the coefficients of $m(X)$, the irreducible representations of B_3 of dimension k are exactly the irreducible representations of $H_k \otimes_\theta K$ of dimension k. A conjecture of Broué, Malle and Rouquier states that H_k is free as R_k-module of rank

[1] Two $k \times k$ matrices are in ordered triangular form if one of them is an upper triangular matrix with eigenvalue λ_i as ith diagonal entry, and the other is a lower triangular matrix with eigenvalue λ_{k+1-i} as ith diagonal entry.

$|W_k|$. Based on this assumption, the irreducible representations of H_k have been determined in [13]. We will show how to use the decomposition map d_θ (see [9] §7.3), in order to get the irreducible representations of $H_k \otimes_\theta K$ that we are interested in.

The general freeness conjecture of Broué, Malle and Rouquier states that the generic Hecke algebra of a complex reflection group is a free R-module of finite rank, where R is the ring of definition of the Hecke algebra (see [4]). For the finite quotients W_k of the braid group we mentioned before, this conjecture is known to be true for the symmetric group (see [9], Lemma 4.4.3), and it was proved in [8,2] and [15] for the case of G_4 and in [15] for the cases of G_{25} and G_{32}. We will prove the validity of the conjecture for the rest of the cases, which belong to the class of complex reflection groups of rank two[2]; the main theorem of this paper is the following:

Theorem 1.1. *H_k is a free R_k-module of rank $|W_k|$.*

By general arguments (see, for example, [16]) this has for consequence the following:

Corollary 1.2. *If F is a suitably large extension of the field of fractions of R_k, then $H_k \otimes_{R_k} F$ is isomorphic to the group algebra FW_k.*

In order to prove this theorem we need some preliminary results, which contain a lot of calculations between the images of some elements of the braid group inside the Hecke algebra. We hope that this will not discourage the reader to study the proof, since these calculations are not that complicated and they should be fairly easy to follow.

2. Preliminaries

Let B_3 be the braid group on 3 strands, given by generators the braids s_1 and s_2 and the single relation $s_1s_2s_1 = s_2s_1s_2$, that we call braid relation.

We set $R_k = \mathbb{Z}[a_{k-1}, ..., a_1, a_0, a_0^{-1}]$, for $k = 2, 3, 4, 5$. Let H_k denote the quotient of the group algebra R_kB_3 by the relations

$$s_i^k = a_{k-1}s_i^{k-1} + ... + a_1s_i + a_0, \tag{1}$$

for $i = 1, 2$. For $k = 2, 3, 4$ and 5 we call H_k the quadratic, cubic, quartic and quintic Hecke algebra, respectively.

We identify s_i to their images in H_k. We multiply (1) by s_i^{-k} and since a_0 is invertible in R_k we have:

$$s_i^{-k} = -a_0^{-1}a_1s_i^{-k+1} - a_0^{-1}a_2s_i^{-k+2} - ... - a_0^{-1}a_{k-1}s_i^{-1} + a_0^{-1}, \tag{2}$$

[2] The study of the conjecture of these groups is the subject of the author's PhD thesis, that is still in progress.

for $i = 1, 2$. If we multiply (2) with a suitable power of s_i we can expand s_i^{-n} as a linear combination of $s_i^{-n+1}, \dots, s_i^{-n+(k-1)}$, s_i^{-n+k}, for every $n \in \mathbb{N}$. Moreover, comparing (1) and (2), we can define an automorphism Φ of H_k as $\mathbb{Z}$-algebra, where

$$\begin{aligned} s_i &\mapsto s_i^{-1}, \text{ for } i = 1, 2 \\ a_j &\mapsto -a_0^{-1} a_{k-j}, \text{ for } j = 1, ..., k-1 \\ a_0 &\mapsto a_0^{-1} \end{aligned}$$

We will prove now an easy lemma that plays an important role in the sequel. This lemma is in fact a generalization of lemma 2.1 of [15].

Lemma 2.1. *For every $m \in \mathbb{Z}$ we have $s_2 s_1^m s_2^{-1} = s_1^{-1} s_2^m s_1$ and $s_2^{-1} s_1^m s_2 = s_1 s_2^m s_1^{-1}$.*

Proof. By using the braid relation we have that $(s_1 s_2) s_1 (s_1 s_2)^{-1} = s_2$. Therefore, for every $m \in \mathbb{Z}$ we have $(s_1 s_2) s_1^m (s_1 s_2)^{-1} = s_2^m$, that gives us the first equality. Similarly, we prove the second one. □

If we assume m of Lemma 2.1 to be positive we have $s_1 s_2 s_1^n = s_2^n s_1 s_2$ and $s_1^n s_2 s_1 = s_2 s_1 s_2^n$, where $n \in \mathbb{N}$. Taking inverses, we also get $s_1^{-n} s_2^{-1} s_1^{-1} = s_2^{-1} s_1^{-1} s_2^{-n}$ and $s_1^{-1} s_2^{-1} s_1^{-n} = s_1^{-n} s_2^{-1} s_1^{-1}$. We call all the above relations *the generalized braid relations.*

We denote by u_i the R_k-subalgebra of H_k generated by s_i (or equivalently by s_i^{-1}) and by $u_i^\times$ the group of units of u_i, where $i = 1, 2$. We also set $\omega := s_2 s_1^2 s_2$. Since the center of B_3 is the subgroup generated by the element $z = s_1^2 \omega$ (see, for example, Theorem 1.24 of [10]), for all $x \in u_1$ and $m \in \mathbb{Z}$ we have that $x\omega^m = \omega^m x$. We will see later that ω plays an important role in the description of H_k.

Let W_k be the quotient group $B_3/\langle s_i^k \rangle$, $k = 2, 3, 4$ and 5. From a Coxeter's theorem (see §10 in [7]) we know that W_k is finite. Let r_k denote the order of W_k. Our goal now is to prove that H_k is a free R_k-module of rank r_k, a statement that holds for H_2 since $W_2 = \mathfrak{S}_3$ is a Coxeter group (see [9], Lemma 4.4.3). For the remaining cases, we will use the following proposition.

Proposition 2.2. *Let $k \in \{3, 4, 5\}$. If H_k is generated as a module over R_k by r_k elements, then H_k is a free R_k-module of rank r_k.*

Proof. The algebras H_k are the generic Hecke algebras of the complex reflection groups G_4, G_8 and G_{16}, respectively in the sense of Broué, Malle and Rouquier (see [4]). Hence, the result follows from Theorem 4.24 in [4] or from Proposition 2.4(1) in [16]. □

Therefore, we need to find a spanning set of H_k, $k = 3, 4, 5$ of r_k elements. However, for the cubic Hecke algebra H_3 we have that $H_3 = u_1 + u_1 s_2 u_1 + u_1 s_2^{-1} u_1 + u_1 s_2 s_1^{-1} s_2$ (see [15], Theorem 3.2(3)) and, hence, H_3 is spanned as a left u_1-module by 8 elements. Since u_1 is spanned by 3 elements as R_3-module, we have that H_3 is spanned over R_3 by $r_3 = 24$ elements.

3. The quartic Hecke algebra H_4

Our ring of definition is $R_4 = \mathbb{Z}[a,b,c,d,d^{-1}]$ and therefore, relation (1) becomes $s_i^4 = as_i^3 + bs_i^2 + cs_i + d$, for $i = 1, 2$. We set

$$U' = u_1u_2u_1 + u_1s_2s_1^{-1}s_2u_1 + u_1s_2^{-1}s_1s_2^{-1}u_1 + u_1s_2^{-1}s_1^{-2}s_2^{-1}$$
$$U = U' + u_1s_2s_1^{-2}s_2u_1 + u_1s_2^{-2}s_1^{-2}s_2^{-2}u_1.$$

It is obvious that U is a u_1-bimodule and that U' is a u_1-sub-bimodule of U. Before proving our main theorem (Theorem 3.3) we need a few preliminaries results.

Lemma 3.1. *For every $m \in \mathbb{Z}$ we have*

(i) $s_2s_1^m s_2 \in U$.
(ii) $s_2^{-1}s_1^m s_2^{-1} \in U'$.
(iii) $s_2^{-2}s_1^m s_2^{-1} \in U'$.

Proof. By using the relations (1) and (2) we can assume that $m \in \{0, 1, -1, -2\}$. Hence, we only have to prove (iii), since (i) and (ii) follow from the definition of U and U' and the braid relation. For (iii), we can assume that $m \in \{-2, 1\}$, since the case where $m = -1$ is obvious by using the generalized braid relations. We have: $s_2^{-2}s_1^{-2}s_2^{-1} = s_1^{-1}(s_1s_2^{-2}s_1^{-1})s_1^{-1}s_2^{-1} = s_1^{-1}s_2^{-1}s_1^{-2}(s_2s_1^{-1}s_2^{-1}) = s_1^{-1}(s_2^{-1}s_1^{-3}s_2^{-1})s_1$. The result then follows from (ii). For the element $s_2^{-2}s_1s_2^{-1}$, we expand s_2^{-2} as a linear combination of s_2^{-1}, 1, s_2, s_2^2 and by using the definition of U' and Lemma 2.1, we only have to check that $s_2^2s_1s_2^{-1} \in U'$. Indeed, we have: $s_2^2s_1s_2^{-1} = s_2(s_2s_1s_2^{-1}) = (s_2s_1^{-1}s_2)s_1 \in U'$. □

Proposition 3.2. $u_2u_1u_2 \subset U$.

Proof. We need to prove that every element of the form $s_2^\alpha s_1^\beta s_2^\gamma$ belongs to U, for $\alpha, \beta, \gamma \in \{-2, -1, 0, 1\}$. However, when $\alpha\beta\gamma = 0$ the result is obvious. Therefore, we can assume $\alpha, \beta, \gamma \in \{-2, -1, 1\}$. We have the following cases:

- $\underline{\alpha = 1}$: The cases where $\gamma \in \{-1, 1\}$ follow from Lemmas 2.1 and 3.1(i). Hence, we need to prove that $s_2s_1^\beta s_2^{-2} \in U$. For $\beta = -1$ we use Lemma 2.1 and we have $s_2s_1^{-1}s_2^{-2} = (s_2s_1^{-1}s_2^{-1})s_2^{-1} = s_1^{-1}(s_2^{-1}s_1s_2^{-1}) \in U$. For $\beta = 1$ we expand s_2^{-2} as a linear combination of s_2^{-1}, 1, s_2, s_2^2 and the result follows from the cases where $\gamma \in \{-1, 0, 1\}$ and the generalized braid relations. It remains to prove that $s_2s_1^{-2}s_2^{-2} \in U$. By expanding now s_1^{-2} as a linear combination of s_1^{-1}, 1, s_1, s_1^2 we only need to prove that $s_2s_1^2s_2^{-2} \in U$ (the rest of the cases correspond to $b = -1$, $b = 0$ and $b = 1$). We use Lemma 2.1 and we have: $s_2s_1^2s_2^{-2} = (s_2s_1^2s_2^{-1})s_2^{-1} = s_1^{-1}s_2(s_2s_1s_2^{-1}) = s_1^{-1}(s_2s_1^{-1}s_2)s_1 \in U$.

- $\underline{\alpha = -1}$: Exactly as in the case where $\alpha = 1$, we only have to prove that $s_2^{-1}s_1^{\beta}s_2^{-2} \in U$. For $\beta = -1$ the result is obvious by using the generalized braid relations. For $\beta = -2$ we have: $s_2^{-1}s_1^{-2}s_2^{-2} = (s_2^{-1}s_1^{-2}s_2)s_2^{-3} = s_1s_2^{-1}(s_2^{-1}s_1^{-1}s_2^{-3}) = s_1(s_2^{-1}s_1^{-3}s_2^{-1})s_1^{-1}$. However, by Lemma 3.1(ii) we have that the element $s_2^{-1}s_1^{-3}s_2^{-1}$ is inside U' and, hence, inside U. It remains to prove that $s_2^{-1}s_1s_2^{-2} \in U$. For this purpose, we expand s_2^{-2} as a linear combination of s_2^{-1}, 1, s_2, s_2^2 and by the definition of U and Lemma 2.1 we only need to prove that $s_2^{-1}s_1s_2^2 \in U$. Indeed, using Lemma 2.1 again we have: $s_2^{-1}s_1s_2^2 = (s_2^{-1}s_1s_2)s_2 = s_1(s_2s_1^{-1}s_2) \in U$.
- $\underline{\alpha = -2}$: We can assume that $\gamma \in \{1, -2\}$, since the case where $\gamma = -1$ follows immediately from Lemma 3.1(iii). For $\gamma = 1$ we use Lemma 2.1 and we have $s_2^{-2}s_1^{\beta}s_2 = s_2^{-1}(s_2^{-1}s_1^{\beta}s_2) = (s_2^{-1}s_1s_2^{\beta})s_1^{-1}$. The latter is an element in U, as we proved in the case where $\alpha = -1$. For $\gamma = -2$ we only need to prove the cases where $\beta = \{-1, 1\}$, since the case where $\beta = -2$ follows from the definition of U. We use the generalized braid relations and we have $s_2^{-2}s_1^{-1}s_2^{-2} = (s_2^{-2}s_1^{-1}s_2^{-1})s_2^{-1} = s_1^{-1}(s_2^{-1}s_1^{-2}s_2^{-1}) \in U$. Moreover, $s_2^{-2}s_1s_2^{-2} = s_1(s_1^{-1}s_2^{-2}s_1)s_2^{-2} = s_1(s_2s_1^{-2}s_2^{-3})$. The result follows from the case where $\alpha = 1$, if we expand s_2^{-3} as a linear combination of s_2^{-2}, s_2^{-1}, 1 and s_2. □

We can now prove the main theorem of this section.

Theorem 3.3.

(i) $U = u_1u_2u_1 + u_1s_2s_1^{-1}s_2u_1 + u_1s_2^{-1}s_1s_2^{-1}u_1 + u_1\omega + u_1\omega^{-1} + u_1\omega^{-2}$.

(ii) $H_4 = U$.

Proof.

(i) We recall that $\omega = s_2s_1^2s_2$. We must prove that the RHS, which is by definition $U' + u_1\omega + u_1\omega^{-2}$, is equal to U. For this purpose we will “replace” inside the definition of U the elements $s_2s_1^{-2}s_2$ and $s_2^{-2}s_1^{-2}s_2^{-2}$ with the elements ω and ω^{-2} modulo U', by proving that $s_2s_1^{-2}s_2 \in u_1^{\times}\omega + U'$ and $s_2^{-2}s_1^{-2}s_2^{-2} \in u_1^{\times}\omega^{-2} + U'$.
For the element $s_2s_1^{-2}s_2$, we expand s_1^{-2} as a linear combination of s_1^{-1}, 1, s_1, s_1^2, where the coefficient of s_1^2 is invertible. The result then follows from the definition of U' and the braid relation. For the element $s_2^{-2}s_1^{-2}s_2^{-2}$ we apply Lemma 2.1 and the generalized braid relations and we have: $s_2^{-2}s_1^{-2}s_2^{-2} = s_2^{-2}s_1^{-1}(s_1^{-1}s_2^{-2}s_1)s_1^{-1} = s_2^{-1}(s_2^{-1}s_1^{-1}s_2)s_1^{-1}(s_1^{-1}s_2^{-1}s_1)s_1^{-2} = s_2^{-1}s_1(s_2^{-1}s_1^{-2}s_2)s_1^{-1}s_2^{-1}s_1^{-2} \in s_2^{-1}s_1^2s_2^{-2}s_1^{-2}s_2^{-1}u_1$. We expand s_1^2 as a linear combination of s_1, 1, s_1^{-1}, s_1^{-2}, where the coefficient of s_1^{-2} is invertible and by the generalized braid relations and the fact that $s_1^{-2}\omega^{-2} = \omega^{-2}s_1^{-2} = s_2^{-1}s_1^{-2}s_2^{-2}s_1^{-2}s_2^{-1}s_1^{-2}$ we have that

$$s_2^{-2}s_1^{-2}s_2^{-2} \in s_2^{-1}s_1s_2^{-2}s_1^{-2}s_2^{-1}u_1 + s_2^{-3}s_1^{-2}s_2^{-1}u_1 + u_1s_2^{-1}s_1^{-3}s_2^{-1}u_1 + u_1^{\times}\omega^{-2}.$$

Therefore, by Lemma 3.1(ii) it is enough to prove that the elements $s_2^{-1}s_1s_2^{-2}s_1^{-2}s_2^{-1}$ and $s_2^{-3}s_1^{-2}s_2^{-1}$ belong to U'. However, the latter is an element in U', if we expand s_2^{-3} as a linear combination of s_2^{-2}, s_2^{-1}, 1, s_2 and use Lemma 3.1(iii), the definition of U' and Lemma 2.1. Moreover, $s_2^{-1}s_1s_2^{-2}s_1^{-2}s_2^{-1}=s_2^{-2}(s_2s_1s_2^{-1})\omega^{-1}=s_2^{-2}s_1^{-1}s_2s_1\omega^{-1}=s_2^{-2}s_1^{-1}s_2\omega^{-1}s_1^{-1}=(s_2^{-2}s_1^{-4}s_2^{-1})s_1^{-1}\in U'$, by Lemma 3.1(iii).

(ii) Since $1\in U$, it will be sufficient to show that U is a left ideal of H_4. We know that U is a u_1-sub-bimodule of H_4. Therefore, we only need to prove that $s_2U\subset U$. Since U is equal to the RHS of (i) we have that

$$s_2U\subset s_2u_1u_2u_1+s_2u_1s_2s_1^{-1}s_2u_1+s_2u_1s_2^{-1}s_1s_2^{-1}u_1+s_2u_1\omega+s_2u_1\omega^{-1}+s_2u_1\omega^{-2}.$$

However, $s_2u_1u_2u_1+s_2u_1\omega+s_2u_1\omega^{-1}+s_2u_1\omega^{-2}=s_2u_1u_2u_1+s_2\omega u_1+s_2\omega^{-1}u_1+s_2\omega^{-2}u_1=s_2u_1u_2u_1+s_2^3s_1^2s_2u_1+s_1^{-2}s_2^{-1}u_1+s_1^{-2}s_2^{-2}s_1^{-1}s_2^{-1}\subset u_1u_2u_1u_2u_1$. Furthermore, by using Lemma 2.1 we have that $s_2u_1s_2^{-1}=s_1^{-1}u_2s_1$. Hence, $s_2u_1s_2s_1^{-1}s_2u_1=(s_2u_1s_2^{-1})s_2^2s_1^{-1}s_2u_1=s_1^{-1}u_2(s_1s_2^2s_1^{-1})s_2u_1=s_1^{-1}u_2s_1^2s_2^2u_1\subset u_1u_2u_1u_2u_1$. Moreover, by using Lemma 2.1 again we have that $(s_2u_1s_2^{-1})s_1s_2^{-1}u_1=s_1^{-1}u_2s_1^2s_2^{-1}u_1\subset u_1u_2u_1u_2u_1$. Therefore,

$$\begin{aligned}&s_2u_1u_2u_1+s_2u_1s_2s_1^{-1}s_2u_1+s_2u_1s_2^{-1}s_1s_2^{-1}u_1+s_2u_1\omega+s_2u_1\omega^{-1}+s_2u_1\omega^{-2}\\&\quad\subset u_1u_2u_1u_2u_1.\end{aligned}$$

The result follows directly from Proposition 3.2. □

Corollary 3.4. *H_4 is a free R_4-module of rank $r_4=96$.*

Proof. By Proposition 2.2 it will be sufficient to show that H_4 is generated as R_4-module by $r_4=96$ elements. By Theorem 3.3 and the fact that $u_1u_2u_1=u_1(R_4+R_4s_2+R_4s_2^{-1}+R_4s_2^2)u_1=u_1+u_1s_2u_1+u_1s_2^{-1}u_1+u_1s_2^2u_1$ we have that H_4 is generated as left u_1-module by 24 elements. Since u_1 is generated by 4 elements as a R_4-module, we have that H_4 is generated over R_4 by 96 elements. □

4. The quintic Hecke algebra H_5

Our ring of definition is $R_5=\mathbb{Z}[a,b,c,d,e,e^{-1}]$ and therefore, relation (1) becomes $s_i^5=as_i^4+bs_i^3+cs_i^2+ds_i+e$, for $i=1,2$. We recall that $\omega=s_2s_1^2s_2$ and we set

$$\begin{aligned}U'=&\,u_1u_2u_1+u_1\omega+u_1\omega^{-1}+u_1s_2^{-1}s_1^2s_2^{-1}u_1+u_1s_2s_1^{-2}s_2u_1+u_1s_2^2s_1^2s_2^2u_1+\\&+u_1s_2^{-2}s_1^{-2}s_2^{-2}u_1+u_1s_2s_1^{-2}s_2^2u_1+u_1s_2^{-1}s_1^2s_2^{-2}u_1+u_1s_2^{-1}s_1s_2^{-1}u_1+\\&+u_1s_2s_1^{-1}s_2u_1+u_1s_2^{-2}s_1^{-2}s_2^2u_1+u_1s_2^2s_1^2s_2^{-2}u_1+u_1s_2^2s_1^{-2}s_2^2u_1+\\&+u_1s_2^{-2}s_1^2s_2^{-2}u_1+u_1s_2^{-2}s_1s_2^{-1}u_1+u_1s_2^{-1}s_1s_2^{-2}u_1\\U''=&\,U'+u_1\omega^2+u_1\omega^{-2}+u_1s_2^{-2}s_1^2s_2^{-1}s_1s_2^{-1}u_1+u_1s_2^2s_1^{-2}s_2s_1^{-1}s_2u_1+\end{aligned}$$

$$\begin{aligned}
&+ u_1 s_2 s_1^{-2} s_2^2 s_1^{-2} s_2^2 u_1 + u_1 s_2^{-1} s_1^2 s_2^{-2} s_1^2 s_2^{-2} u_1\\
U''' &= U'' + u_1\omega^3 + u_1\omega^{-3}\\
U'''' &= U''' + u_1\omega^4 + u_1\omega^{-4}\\
U &= U'''' + u_1\omega^5 + u_1\omega^{-5}.
\end{aligned}$$

It is obvious that U is a u_1-bi-module and that U', U'', U''' and U'''' are u_1- sub-bi-modules of U. Again, our goal is to prove that $H_5 = U$ (Theorem 4.10). As we explained in the proof of Theorem 3.3, since $1 \in U$ and $s_1 U \subset U$ (by the definition of U), it is enough to prove that $s_2 U \subset U$. We notice that

$$U = \sum_{k=1}^{5} u_1\omega^{\pm k} + \underbrace{u_1 u_2 u_1 + u_1\text{“some elements of length 3”}u_1}_{\in U'} + \\ + \underbrace{u_1\text{“some elements of length 5”}u_1}_{\in U''}.$$

By the definition of U' and U'' we have that $u_1\omega^{\pm 1} \subset U'$ and $u_1\omega^{\pm 2} \subset U''$. Therefore, in order to prove that $s_2 U \subset U$ we only need to prove that $s_2 u_1 \omega^{\pm k}$ $(k = 3, 4, 5)$, $s_2 U'$ and $s_2 U''$ are subsets of U.

The rest of this section is devoted to this proof (see Proposition 4.6, Lemma 4.3(ii), Proposition 4.8(i), (ii) and Theorem 4.10). The reason we define also U''' and U'''' is because, in order to prove that $s_2 u_1 \omega^k$ and $s_2 u_1 \omega^{-k}$ $(k = 3, 4, 5)$ are subsets of U, we want to “replace” inside the definition of U the elements ω^k and ω^{-k} by some other elements modulo U'', U''' and U'''', respectively (see Lemmas 4.5, 4.7 and 4.9).

Recalling that Φ is the automorphism of H_5 as defined in section 2, we have the following lemma:

Lemma 4.1. *The u_1-bi-modules U', U'', U''', U'''' and U are stable under Φ.*

Proof. We notice that U', U'', U''', U'''' and U are of the form

$$u_1 s_2^{-2} s_1 s_2^{-1} u_1 + u_1 s_2^{-1} s_1 s_2^{-2} u_1 + \sum u_1 \sigma u_1 + \sum u_1 \sigma^{-1} u_1,$$

for some $\sigma \in B_3$ satisfying $\sigma^{-1} = \Phi(\sigma)$ and $\sigma = \Phi(\sigma^{-1})$. Therefore, we restrict ourselves to proving that the elements $\Phi(s_2^{-2} s_1 s_2^{-1}) = s_2^2 s_1^{-1} s_2$ and $\Phi(s_2^{-1} s_1 s_2^{-2}) = s_2 s_1^{-1} s_2^2$ belong to U'. We expand s_2^2 as a linear combination of s_2, 1, s_2^{-1}, s_2^{-2} and s_2^{-3} and by the definition of U' and Lemma 2.1 we have to prove that the elements $s_2^k s_1^{-1} s_2$ and $s_2 s_1^{-1} s_2^k$ are elements in U', for $k = -3, -2$. Indeed, by using Lemma 2.1 we have: $s_2^k s_1^{-1} s_2 = s_2^{k+1}(s_2^{-1} s_1^{-1} s_2) = (s_2^{k+1} s_1 s_2^{-1}) s_1^{-1} \in U'$ and $s_2 s_1^{-1} s_2^k = (s_2 s_1^{-1} s_2^{-1}) s_2^{k+1} = s_1^{-1}(s_2^{-1} s_1 s_2^{k+1}) \in U'$. □

From now on, we will use Lemma 2.1 without mentioning it.

Proposition 4.2. $u_2u_1u_2 \subset U'$.

Proof. We have to prove that every element of the form $s_2^{\alpha}s_1^{\beta}s_2^{\gamma}$ belongs to U', for $\alpha, \beta, \gamma \in \{-2, -1, 0, 1, 2\}$. However, when $\alpha\beta\gamma = 0$ the result is obvious. Therefore, we can assume that $\alpha, \beta, \gamma \in \{-2, -1, 1, 2\}$. We continue the proof as in the proof of Proposition 3.2, which is by distinguishing cases for α. However, by using Lemma 4.1 we can assume that $\alpha \in \{1, 2\}$. We have:

- $\underline{\alpha = 1}$:
  - $\underline{\gamma \in \{-1, 1\}}$: The result follows from Lemma 2.1, the braid relation and the definition of U'.
  - $\underline{\gamma = -2}$: $s_2s_1^{\beta}s_2^{-2} = (s_2s_1^{\beta}s_2^{-1})s_2^{-1} = s_1^{-1}(s_2^{\beta}s_1s_2^{-1})$. For $\beta \in \{1, -1, -2\}$ the result follows from Lemma 2.1 and the definition of U'. For $\beta = 2$, we have $s_1^{-1}s_2^2s_1s_2^{-1} = s_1^{-1}s_2(s_2s_1s_2^{-1}) = s_1^{-1}(s_2s_1^{-1}s_2)s_1 \in U'$.
  - $\underline{\gamma = 2}$: We need to prove that the element $s_2s_1^{\beta}s_2^2$ is inside U'. For $\beta \in \{-2, 1\}$ the result is obvious by using the definition of U' and the generalized braid relations. For $\beta = -1$ we have $s_2s_1^{-1}s_2^2 = \Phi(s_2^{-1}s_1s_2^{-2}) \in \Phi(U') \overset{4.1}{=} U'$. For $\beta = 2$ we have $s_2s_1^2s_2^2 = s_1^{-1}(s_1s_2s_1^2)s_2^2 = s_1^{-1}s_2(s_2s_1s_2^3) = s_1^{-1}(s_2s_1^3s_2)s_1$. The result then follows from the case where $\gamma = 1$, if we expand s_1^3 as a linear combination of s_1^2, s_1, 1, s_1^{-1}, s_1^{-2}.
- $\underline{\alpha = 2}$:
  - $\underline{\gamma = -1}$: $s_2^2s_1^{\beta}s_2^{-1} = s_2(s_2s_1^{\beta}s_2^{-1}) = (s_2s_1^{-1}s_2^{\beta})s_1 \in U'$ (case where $\alpha = 1$).
  - $\underline{\gamma = 2}$: We only have to prove the cases where $\beta \in \{-1, 1\}$, since the cases where $\beta \in \{2, -2\}$ follow from the definition of U'. We have $s_2^2s_1s_2^2 = (s_2^2s_1s_2)s_2 = s_1\omega \in U'$. Moreover, $s_2^2s_1^{-1}s_2^2 = s_1^{-1}(s_1s_2^2s_1^{-1})s_2^2 = s_1^{-1}\Phi(s_2s_1^{-2}s_2^{-3})$. The result follows from the case where $\alpha = 1$ and Lemma 4.1, if we expand s_2^{-3} as a linear combination of s_2^{-2}, s_2^{-1}, 1, s_2, s_2^2.
  - $\underline{\gamma = 1}$: We have to check the cases where $\beta \in \{-2, -1, 2\}$, since the case where $\beta = 1$ is a direct result from the generalized braid relations. However, $s_2^2s_1^{-1}s_2 = \Phi(s_2^{-2}s_1s_2^{-1}) \in \Phi(U') \overset{4.1}{=} U'$. Hence, it remains to prove the cases where $\beta \in \{-2, 2\}$. We have $s_2^2s_1^{-2}s_2 = s_2^3(s_2^{-1}s_1^{-2}s_2) = s_1(s_1^{-1}s_2^3s_1)s_2^{-2}s_1^{-1} = s_1(s_2s_1^3s_2^{-3})s_1^{-1}$. The latter is an element in U', if we expand s_1^3 and s_2^{-3} as linear combinations of s_1^2, s_1, 1, s_1^{-1}, s_1^{-2} and s_2^{-2}, s_2^{-1}, 1, s_2, s_2^2, respectively and use the case where $\alpha = 1$. Moreover, $s_2^2s_1^2s_2 = s_2^2s_1(s_1s_2s_1)s_1^{-1} = (s_2^2s_1s_2)s_1s_2s_1^{-1} = s_1(s_2s_1^3s_2)s_1$. The result follows again from the case where $\alpha = 1$, if we expand s_1^3 as a linear combination of s_1^2, s_1, 1, s_1^{-1}, s_1^{-2}.
  - $\underline{\gamma = -2}$: We need to prove that $s_2^2s_1^{\beta}s_2^{-2} \in U'$. For $\beta = 2$ the result follows from the definition of U'. For $\beta \in \{1, -1\}$ we have: $s_2^2s_1s_2^{-2} = s_2^2(s_1s_2^{-2}s_1^{-1})s_1 = (s_2s_1^{-2}s_2)s_1 \in U'$. $s_2^2s_1^{-1}s_2^{-2} = s_2(s_2s_1^{-1}s_2^{-1})s_2^{-1} = (s_2s_1^{-1}s_2^{-1})s_1s_2^{-1} = s_1^{-1}(s_2^{-1}s_1^2s_2^{-1}) \in U'$. It remains to prove the case where $\beta = -2$. We recall that $\omega = s_2s_1^2s_2$ and we have: $s_2^2s_1^{-2}s_2^{-2} = s_1^{-1}(s_1s_2^2s_1^{-1})s_1^{-1}s_2^{-2} = s_1^{-1}s_2^{-2}\omega s_1^{-1}s_2^{-2} =$

$s_1^{-1}s_2^{-2}s_1^{-1}\omega s_2^{-2} = s_1^{-1}s_2^{-2}s_1^{-1}(s_2s_1^2s_2^{-1}) = s_1^{-1}(s_2^{-2}s_1^{-2}s_2^2)s_1$. The result follows from the definition of U'. □

From now on, in order to make it easier for the reader to follow the calculations, we will underline the elements belonging to $u_1u_2u_1u_2u_1$ and we will use immediately the fact that these elements belong to U' (see Proposition 4.2).

Lemma 4.3.

(i) $s_2u_1s_2u_1s_2u_1 \subset \omega^2u_1 + u_1u_2u_1u_2u_1 \subset U''$.
(ii) $s_2\omega^2u_1 = s_1s_2s_1^4s_2s_1^3s_2u_1 \subset U''$.

Proof. We recall that $\omega = s_2s_1^2s_2$.

(i) The fact that $\omega^2u_1 + u_1u_2u_1u_2u_1 \subset U''$ follows directly from the definition of U'' and Proposition 4.2. For the rest of the proof, we use the definition of u_1 and we have that $s_2u_1s_2u_1s_2u_1 = s_2u_1s_2(R_5 + R_5s_1^{-1} + R_5s_1 + R_5s_1^2 + R_5s_1^3)s_2u_1 \subset \underline{s_2u_1s_2^2u_1} + s_2u_1s_2s_1^{-1}s_2u_1 + \underline{s_2u_1(s_2s_1s_2)u_1} + s_2u_1\omega + s_2u_1s_2s_1^3s_2u_1$. However, $s_2u_1\omega = \underline{s_2\omega u_1}$ and $s_2u_1s_2s_1^{-1}s_2u_1 = s_2u_1(s_1s_2s_1^{-1})s_2u_1 = (s_2u_1s_2^{-1})s_1s_2^2u_1 = \underline{s_1^{-1}u_2s_1^2s_2^2u_1}$. Therefore, it is enough to prove that $s_2u_1s_2s_1^3s_2u_1 \subset \omega^2u_1 + u_1u_2u_1u_2u_1$. For this purpose, we use again the definition of u_1 and we have:

$$\begin{aligned} s_2u_1s_2s_1^3s_2u_1 &\subset s_2(R_5 + R_5s_1 + R_5s_1^{-1} + R_5s_1^2 + R_5s_1^3)s_2s_1^3s_2u_1 \\ &\subset \underline{s_2^2s_1^3s_2u_1} + \underline{s_2(s_1s_2s_1^3)s_2u_1} + s_2(s_1^{-1}s_2s_1)s_1^2s_2u_1 + \omega s_1^3s_2u_1 + \\ &\quad + s_2s_1^2(s_1s_2s_1^3)s_2u_1 \\ &\subset \underline{s_2^2s_1(s_2^{-1}s_1^2s_2)u_1} + \underline{s_1^3\omega s_2u_1} + s_2s_1^2s_2^2(s_2s_1s_2^2)u_1 + u_1u_2u_1u_2u_1 \\ &\subset \omega^2u_1 + u_1u_2u_1u_2u_1. \end{aligned}$$

(ii) We have that $s_2\omega^2 = s_1(s_1^{-1}s_2^2s_1)(s_1s_2s_1)s_1^{-1}\omega = s_1s_2s_1^4(s_1^{-1}s_2s_1)s_1^{-2}\omega = s_1s_2s_1^4s_2s_1s_2^{-1}s_1^{-2}\omega = s_1s_2s_1^4s_2s_1s_2^{-1}\omega s_1^{-2} = s_1s_2s_1^4s_2s_1^3s_2s_1^{-2}$. Therefore, $s_2\omega^2u_1 \subset u_1s_2u_1s_2u_1s_2u_1$. The fact that $u_1s_2u_1s_2u_1s_2u_1 \subset U''$ follows immediately from (i). □

Proposition 4.4.

(i) $u_2u_1s_2^{-1}s_1s_2^{-1} \subset u_1\omega^{-2} + R_5s_2^{-2}s_1^2s_2^{-1}s_1s_2^{-1} + u_1u_2u_1u_2u_1 \subset U''$.
(ii) $u_2u_1s_2s_1^{-1}s_2 \subset u_1\omega^2 + R_5s_2^2s_1^{-2}s_2s_1^{-1}s_2 + u_1u_2u_1u_2u_1 \subset U''$.

Proof. We restrict ourselves to proving (i), since (ii) follows from (i) by applying Φ (see Lemma 4.1). By the definition of U'' and by Proposition 4.2 we have that

$u_1\omega^{-2}+R_5s_2^{-2}s_1^2s_2^{-1}s_1s_2^{-1}+u_1u_2u_1u_2u_1\subset U''$. Therefore, it remains to prove that $u_2u_1s_2^{-1}s_1s_2^{-1}\subset u_1\omega^{-2}+R_5s_2^{-2}s_1^2s_2^{-1}s_1s_2^{-1}+u_1u_2u_1u_2u_1$.

By he definition of u_1 we have that $u_2u_1s_2^{-1}s_1s_2^{-1}=u_2(R_5+R_5s_1+R_5s_1^{-1}+R_5s_1^{-2}+R_5s_1^2)s_2^{-1}s_1s_2^{-1}\subset \underline{u_2s_1s_2^{-1}}+u_2s_1s_2^{-1}s_1s_2^{-1}+\underline{u_2(s_1^{-1}s_2^{-1}s_1)s_2^{-1}}+u_2s_1^{-2}s_2^{-1}s_1s_2^{-1}+u_2s_1^2s_2^{-1}s_1s_2^{-1}$. We notice that $u_2s_1s_2^{-1}s_1s_2^{-1}=u_2(s_2s_1s_2^{-1})s_1s_2^{-1}=\underline{u_2s_1^{-1}(s_2s_1^2s_2^{-1})}$. Therefore, we only have to prove that $u_2s_1^{-2}s_2^{-1}s_1s_2^{-1}$ and $u_2s_1^2s_2^{-1}s_1s_2^{-1}$ are subsets of $u_1\omega^{-2}+R_5s_2^{-2}s_1^2s_2^{-1}s_1s_2^{-1}+u_1u_2u_1u_2u_1$. We have:

$$\begin{aligned}
u_2s_1^{-2}s_2^{-1}s_1s_2^{-1}&\subset(R_5+R_5s_2+R_5s_2^{-1}+R_5s_2^2+R_5s_2^3)s_1^{-2}s_2^{-1}s_1s_2^{-1}\\
&\subset \underline{R_5s_1^{-2}s_2^{-1}s_1s_2^{-1}}+\underline{R_5(s_2s_1^{-2}s_2^{-1})s_1s_2^{-1}}+R_5\omega^{-1}s_1s_2^{-1}+\\
&\quad+R_5s_2(s_2s_1^{-2}s_2^{-1})s_1s_2^{-1}+R_5s_2^2(s_2s_1^{-2}s_2^{-1})s_1s_2^{-1}\\
&\subset \underline{R_5s_1\omega^{-1}s_2^{-1}}+R_5(s_2s_1^{-1}s_2^{-1})s_2^{-1}s_1^2s_2^{-1}+R_5s_2(s_2s_1^{-1}s_2^{-1})s_2^{-1}s_1^2s_2^{-1}+\\
&\quad+u_1u_2u_1u_2u_1\\
&\subset R_5s_1^{-1}s_2^{-1}s_1s_2^{-1}s_1^2s_2^{-1}+R_5(s_2s_1^{-1}s_2^{-1})s_1s_2^{-1}s_1^2s_2^{-1}+\\
&\quad+u_1u_2u_1u_2u_1\\
&\subset\Phi(u_1s_2u_1s_2u_1s_2)+u_1u_2u_1u_2u_1.
\end{aligned}$$

However, by Lemma 4.3(i) we have that $\Phi(u_1s_2u_1s_2u_1s_2)\subset\Phi(\omega^2u_1+u_1u_2u_1u_2u_1)=\omega^{-2}u_1+u_1u_2u_1u_2u_1$. Therefore, $u_2s_1^{-2}s_2^{-1}s_1s_2^{-1}\subset\omega^{-2}u_1+u_1u_2u_1u_2u_1$. By using analogous calculations, we have:

$$\begin{aligned}
u_2s_1^2s_2^{-1}s_1s_2^{-1}&\subset(R_5+R_5s_2+R_5s_2^{-1}+R_5s_2^2+R_5s_2^{-2})s_1^2s_2^{-1}s_1s_2^{-1}\\
&\subset\underline{R_5s_1^2s_2^{-1}s_1s_2^{-1}}+\underline{R_5(s_2s_1^2s_2^{-1})s_1s_2^{-1}}+R_5s_2^{-1}s_1^3(s_1^{-1}s_2^{-1}s_1)s_2^{-1}+\\
&\quad+R_5s_2(s_2s_1^2s_2^{-1})s_1s_2^{-1}+R_5s_2^{-2}s_1^2s_2^{-1}s_1s_2^{-1}\\
&\subset\underline{R_5(s_2^{-1}s_1^3s_2)s_1^{-1}s_2^{-2}}+R_5s_2s_1^{-1}s_2^2s_1^2s_2^{-1}+R_5s_2^{-2}s_1^2s_2^{-1}s_1s_2^{-1}+\\
&\quad+u_1u_2u_1u_2u_1.
\end{aligned}$$

It is enough to prove that $s_2s_1^{-1}s_2^2s_1^2s_2^{-1}\subset u_1u_2u_1u_2u_1$. Indeed, we have: $s_2s_1^{-1}s_2^2s_1^2s_2^{-1}=s_1^{-1}(s_1s_2s_1^{-1})s_2(s_2s_1^2s_2^{-1})=\underline{s_1^{-1}s_2^{-1}(s_1s_2^2s_1^{-1})s_2^2s_1}$. □

We can now prove a lemma that helps us to "replace" inside the definition of U''' the element ω^3 with the element $s_2s_1^3s_2^2s_1^2s_2^2$ modulo U''.

Lemma 4.5. *$s_2s_1^3s_2^2s_1^2s_2^2\in u_1s_2u_1s_2s_1^3s_2u_1+u_1s_2^2s_1^3s_2s_1^{-1}s_2u_1+u_1u_2u_1u_2u_1+u_1^\times\omega^3\subset u_1^\times\omega^3+U''$.*

Proof. The fact that $u_1s_2u_1s_2s_1^3s_2u_1+u_1s_2^2s_1^3s_2s_1^{-1}s_2u_1+u_1u_2u_1u_2u_1+u_1^\times\omega^3$ is a subset of $u_1^\times\omega^3+U''$ follows from Lemma 4.3(i) and Propositions 4.4(ii) and 4.2.

For the rest of the proof, we have: $s_2s_1^3s_2^2s_1^2s_2^2 = s_2s_1^2(s_1s_2^2s_1^{-1})s_1^2(s_1s_2^2s_1^{-1})s_1 = s_2s_1^2s_2^{-2}\omega s_1^2s_2^{-1}s_1(s_1s_2s_1^{-1})s_1^2 = s_2s_1^2s_2^{-2}s_1^2\omega s_2^{-1}s_1s_2^{-1}(s_1s_2s_1^{-1})s_1^3 = s_2s_1^2s_2^{-2}s_1^2s_2s_1^3s_2^{-2}s_1s_2s_1^3 = s_2s_1^2s_2^{-3}\boldsymbol{\omega s_1^3s_2^{-2}}s_1s_2s_1^3$. However, $\boldsymbol{\omega s_1^3s_2^{-2}} = s_1^3\omega s_2^{-2} = s_1^3(s_2s_1^2s_2^{-1}) = s_1^2s_2^2s_1$ and, hence, $s_2s_1^3s_2^2s_1^2s_2^2 = s_2s_1^2s_2^{-3}s_1^2s_2^2s_1^2s_2s_1^3$.

Our goal now is to prove that the element $s_2s_1^2s_2^{-3}s_1^2s_2^2s_1^2s_2s_1^3$ is inside $u_1s_2u_1s_2s_1^3s_2u_1 + u_1s_2^2s_1^3s_2s_1^{-1}s_2u_1 + u_1u_2u_1u_2u_1 + u_1^\times\omega^3$. For this purpose we expand s_2^{-3} as a linear combination of s_2^{-2}, s_2^{-1}, 1, s_2 and s_2^2, where the coefficient of s_2^2 is invertible, and we have that $s_2s_1^2s_2^{-3}s_1^2s_2^2s_1^2s_2s_1^3 \in s_2s_1^2s_2^{-2}s_1^2s_2^2s_1^2s_2u_1 + s_2s_1^2s_2^{-1}s_1^2s_2^2s_1^2s_2u_1 + s_2s_1^4s_2^2s_1^2s_2u_1 + s_2\omega^2u_1 + u_1^\times\omega^3$. However, by Lemma 4.3(ii) we have that $s_2\omega^2u_1 \subset u_1s_2u_1s_2s_1^3s_2u_1$. Moreover, $s_2s_1^4s_2^2s_1^2s_2u_1 = s_2s_1^5(s_1^{-1}s_2^2s_1)(s_1s_2s_1)u_1 \subset u_1s_2u_1s_2s_1^3s_2u_1$. It remains to prove that the elements $s_2s_1^2s_2^{-2}s_1^2s_2^2s_1^2s_2$ and $s_2s_1^2s_2^{-1}s_1^2s_2^2s_1^2s_2$ are inside $u_1s_2u_1s_2s_1^3s_2u_1 + u_1s_2^2s_1^3s_2s_1^{-1}s_2u_1 + u_1u_2u_1u_2u_1$.

On one hand, we have $s_2s_1^2s_2^{-2}s_1^2s_2^2s_1^2s_2 = s_2s_1^3(s_1^{-1}s_2^{-2}s_1)s_1s_2\omega = s_2s_1^3s_2s_1^{-1}(s_1^{-1}s_2^{-1}s_1)s_2\omega = s_2s_1^3s_2^2(s_2^{-1}s_1^{-1}s_2)s_1^{-1}\omega = s_2s_1^3s_2^2s_1s_2^{-1}s_1^{-2}\omega = s_2s_1^3s_2^2s_1s_2^{-1}\omega s_1^{-2} = s_2s_1^3s_2^2s_1^3s_2s_1^{-2}$, meaning that the element $s_2s_1^2s_2^{-2}s_1^2s_2^2s_1^2s_2$ is inside $s_2s_1^3s_2^2u_1s_2u_1$. On the other hand, $s_2s_1^2s_2^{-1}s_1^2s_2^2s_1^2s_2 = s_2s_1^2(s_2^{-1}s_1^2s_2)\omega = s_2s_1^3s_2^2s_1^{-1}\omega = s_2s_1^3s_2^2\omega s_1^{-1} = s_2s_1^3s_2^3s_1^2s_2s_1^{-1}$ and, if we expand s_2^3 as a linear combination of s_2^2, s_2, 1, s_2^{-1} and s_2^{-2}, we have that $s_2s_1^3s_2^3s_1^2s_2s_1^{-1} \in s_2s_1^3s_2^2s_1^2s_2u_1 + s_2s_1^3\omega u_1 + \underline{s_2s_1^5s_2u_1} + \underline{(s_2s_1^3s_2^{-1})s_1^2s_2u_1} + \underline{(s_2s_1^3s_2^{-1})(s_2^{-1}s_1^2s_2)u_1} \subset s_2s_1^3s_2^2u_1s_2u_1 + \underline{s_2\omega u_1} + u_1u_2u_1u_2u_1$, meaning that the element $s_2s_1^2s_2^{-1}s_1^2s_2^2s_1^2s_2$ is inside $s_2s_1^3s_2^2u_1s_2u_1 + u_1u_2u_1u_2u_1$. As a result, in order to finish the proof, it will be sufficient to show that $s_2s_1^3s_2^2u_1s_2u_1$ is a subset of $u_1s_2u_1s_2s_1^3s_2u_1 + u_1s_2^2s_1^3s_2s_1^{-1}s_2u_1 + u_1u_2u_1u_2u_1$. Indeed, we have:

$$\begin{aligned}
s_2s_1^3s_2^2u_1s_2u_1 &\subset s_2s_1^3s_2^2(R_5s_1^2 + R_5s_1 + R_5 + R_5s_1^{-1} + R_5s_1^{-2})s_2u_1\\
&\subset s_2s_1^3s_2^2s_1^2s_2u_1 + \underline{s_2s_1^3(s_2^2s_1s_2)u_1} + \underline{s_2s_1^3s_2^3u_1} + s_2s_1^2(s_1s_2^2s_1^{-1})s_2u_1 +\\
&\quad + s_2s_1^2(s_1s_2^2s_1^{-1})s_1^{-1}s_2u_1\\
&\subset s_2s_1^4(s_1^{-1}s_2^2s_1)(s_1s_2s_1)u_1 + \underline{(s_2s_1^2s_2^{-1})s_1^2s_2^2u_1} + (s_2s_1^2s_2^{-1})s_1^2s_2s_1^{-1}s_2u_1 +\\
&\quad + u_1u_2u_1u_2u_1\\
&\subset u_1s_2u_1s_2s_1^3s_2u_1 + u_1s_2^2s_1^3s_2s_1^{-1}s_2u_1 + u_1u_2u_1u_2u_1. \qquad \square
\end{aligned}$$

Proposition 4.6.

(i) $s_2u_1u_2u_1u_2 \subset U'''$.
(ii) $s_2^{-1}u_1u_2u_1u_2 \subset U'''$.

Proof. By Lemma 4.1, we only have to prove (i), since (ii) is a consequence of (i) up to applying Φ. We know that $u_2u_1u_2 \subset U'$ (Proposition 4.2) hence it is enough to prove that $s_2U' \subset U'''$. Set

$$V = u_1u_2u_1 + \omega u_1 + \omega^{-1}u_1 + u_1s_2^{-1}s_1^2s_2^{-1}u_1 + u_1s_2^{-1}s_1s_2^{-1}u_1 + u_1s_2s_1^{-1}s_2u_1 +$$

$$+ u_1 s_2^{-2} s_1 s_2^{-1} u_1 + u_1 s_2^{-1} s_1^2 s_2^{-2} u_1 + u_1 s_2^{-1} s_1 s_2^{-2} u_1 + u_1 s_2 s_1^{-2} s_2 u_1 +$$
$$+ u_1 s_2^{-2} s_1^{-2} s_2^{-2} u_1 + u_1 s_2^{-2} s_1^{-2} s_2^2 u_1.$$

We notice that

$$U' = V + u_1 s_2 s_1^{-2} s_2^2 u_1 + u_1 s_2^2 s_1^{-2} s_2^2 u_1 + u_1 s_2^2 s_1^2 s_2^2 u_1 + u_1 s_2^{-2} s_1^2 s_2^{-2} u_1 + u_1 s_2^2 s_1^2 s_2^{-2} u_1.$$

Therefore, in order to prove that $s_2 U' \subset U'''$, we will prove first that $s_2 V \subset U'''$ and then we will check the other five cases separately. We have:

$$s_2 V \subset \underline{s_2 u_1 u_2 u_1} + \underline{s_2 \omega u_1} + \underline{s_2 \omega^{-1} u_1} + \underline{(s_2 u_1 s_2^{-1}) u_1 u_2 u_1} + s_2 u_1 s_2 u_1 s_2 +$$
$$+ s_2 u_1 s_2^{-2} s_1 s_2^{-1} s_2 u_1 s_2^{-2} s_1^{-2} s_2^{-2} u_1 + s_2 u_1 s_2^{-2} s_1^{-2} s_2^2 u_1 + U'''$$

However, by Proposition 4.3(i) we have that $s_2 u_1 s_2 u_1 s_2 \subset U'' \subset U'''$. It remains to prove that $A := s_2 u_1 s_2^{-2} s_1 s_2^{-1} + s_2 u_1 s_2^{-2} s_1^{-2} s_2^{-2} u_1 + s_2 u_1 s_2^{-2} s_1^{-2} s_2^2 u_1$ is a subset of U'''. We have:

$$\begin{aligned}
A &= s_2 u_1 s_2^{-2} s_1 s_2^{-1} + s_2 u_1 s_2^{-2} s_1^{-2} s_2^{-2} u_1 + s_2 u_1 s_2^{-2} s_1^{-2} s_2^2 u_1 \\
&= (s_2 u_1 s_2^{-1}) s_2^{-1} s_1 s_2^{-1} u_1 + (s_2 u_1 s_2^{-1}) s_2^{-1} s_1^{-2} s_2^{-2} u_1 + (s_2 u_1 s_2^{-1}) s_2^{-1} s_1^{-2} s_2^2 u_1 \\
&= s_1^{-1} u_2 (s_1 s_2^{-1} s_1^{-1}) s_1^2 s_2^{-1} u_1 + s_1^{-1} u_2 (s_1 s_2^{-1} s_1^{-1}) s_1^{-1} s_2^{-2} u_1 + s_1^{-1} u_2 s_1 (s_2^{-1} s_1^{-2} s_2) s_2 u_1 \\
&= s_1^{-1} u_2 s_1^{-1} (s_2 s_1^2 s_2^{-1}) u_1 + s_1^{-1} u_2 s_1^{-1} (s_2 s_1^{-1} s_2^{-1}) s_2^{-1} u_1 + s_1^{-1} u_2 s_1^2 s_2^{-1} (s_2^{-1} s_1^{-1} s_2) u_1 \\
&\subset u_1 (u_2 u_1 s_2^{-1} s_1 s_2^{-1}) u_1.
\end{aligned}$$

By Proposition 4.4 we have then $A \subset U'''$ and, hence, we proved that

$$s_2 V \subset U''' \tag{3}$$

In order to finish the proof that $s_2 U' \subset U''$, it will be sufficient to prove that $u_1 s_2 s_1^{-2} s_2^2 u_1$, $u_1 s_2^2 s_1^{-2} s_2^2 u_1$, $u_1 s_2^2 s_1^2 s_2^2 u_1$, $u_1 s_2^{-2} s_1^2 s_2^{-2} u_1$ and $u_1 s_2^2 s_1^2 s_2^{-2} u_1$ are subsets of U'''.

C1. We will prove that $s_2 u_1 s_2 s_1^{-2} s_2^2 u_1 \subset U'''$. We expand s_2^2 as a linear combination of s_2, 1 s_2^{-1}, s_2^{-2} and s_2^{-3} and we have that $s_2 u_1 s_2 s_1^{-2} s_2^2 u_1 \subset s_2 u_1 s_2 s_1^{-2} s_2 u_1 + \underline{s_2 u_1 s_2 u_1} + \underline{s_2 u_1 (s_2 s_1^{-2} s_2^{-1}) u_1} + s_2 u_1 (s_2 s_1^{-2} s_2^{-1}) s_2^{-1} u_1 + s_2 u_1 s_2 s_1^{-2} s_2^{-3} u_1 \subset s_2 u_1 s_2 s_1^{-2} s_2^{-3} u_1 + \underline{s_2 V + U'''}$ and, hence, by relation (3) we have that $s_2 u_1 s_2 s_1^{-2} s_2^2 u_1 \subset s_2 u_1 s_2 s_1^{-2} s_2^{-3} u_1 + U'''$. Therefore, it will be sufficient to prove that $s_2 u_1 s_2 s_1^{-2} s_2^{-3} u_1 \subset U'''$. We use the definition of u_1 and we have:

$$\begin{aligned}
s_2 u_1 s_2 s_1^{-2} s_2^{-3} u_1 &\subset s_2 (R_5 + R_5 s_1 + R_5 s_1^{-1} + R_5 s_1^2 + R_5 s_1^3) s_2 s_1^{-2} s_2^{-3} u_1 \\
&\subset \underline{s_2^2 s_1^{-2} s_2^{-3} u_1} + \underline{(s_2 s_1 s_2) s_1^{-2} s_2^{-3} u_1} + s_2 s_1^{-1} s_2 s_1^{-2} s_2^{-3} u_1 + \\
&\quad + \omega s_1^{-2} s_2^{-3} u_1 + s_2 s_1^3 s_2 s_1^{-2} s_2^{-3} u_1
\end{aligned}$$

$$\subset s_1^{-1}(s_1s_2s_1^{-1})s_2s_1^{-2}s_2^{-3}u_1 + \underline{s_1^{-2}\omega s_2^{-3}u_1} + $$
$$+ s_2s_1^2(s_1s_2s_1^{-1})s_1^{-1}s_2^{-3}u_1 + U'''$$
$$\subset s_1^{-1}s_2^{-1}(s_1s_2^2s_1^{-1})s_1^{-1}s_2^{-3}u_1 + (s_2s_1^2s_2^{-1})s_1s_2s_1^{-1}s_2^{-3}u_1 + U'''$$
$$\subset s_1^{-1}s_2^{-2}s_1^2(s_2s_1^{-1}s_2^{-1})s_2^{-2}u_1 + s_1^{-1}s_2^2s_1(s_1s_2s_1^{-1})s_2^{-3}u_1 + U'''$$
$$\subset s_1^{-1}s_2^{-3}(s_2s_1s_2^{-1})s_1s_2^{-2}u_1 + s_1^{-1}s_2(s_2s_1s_2^{-1})s_1s_2^{-2}u_1 + U'''$$
$$\subset s_1^{-1}s_2^{-3}s_1^{-1}(s_2s_1^2s_2^{-1})s_2^{-1}u_1 + s_1^{-1}s_2s_1^{-1}(s_2s_1^2s_2^{-1})s_2^{-1}u_1 + U'''$$
$$\subset s_1^{-1}s_2^{-3}s_1^{-2}s_2(s_2s_1s_2^{-1})u_1 + s_1^{-1}s_2s_1^{-2}s_2(s_2s_1s_2^{-1})u_1 + U'''$$
$$\subset u_1(u_2u_1s_2s_1^{-1}s_2)u_1 + U'''.$$

The result follows from Proposition 4.4(ii).

C2. We will prove that $s_2u_1s_2^2s_1^{-2}s_2^2u_1 \subset U'''$. For this purpose, we expand u_1 as $R_5 + R_5s_1 + R_5s_1^4 + R_5s_1^2 + R_5s_1^{-2}$ and we have that $s_2u_1s_2^2s_1^{-2}s_2^2u_1 \subset \underline{s_2^3s_1^{-2}s_2^2u_1} + \underline{(s_2s_1s_2^2)s_1^{-2}s_2^2u_1} + s_2s_1^4s_2^2s_1^{-2}s_2^2u_1 + s_2s_1^2s_2^2s_1^{-2}s_2^2u_1 + s_2s_1^{-2}s_2^2s_1^{-2}s_2^2u_1$. By the definition of U''' we have that $s_2s_1^{-2}s_2^2s_1^{-2}s_2^2u_1 \subset U'''$. Therefore, it remains to prove that $s_2s_1^4s_2^2s_1^{-2}s_2^2u_1 + s_2s_1^2s_2^2s_1^{-2}s_2^2u_1 \subset U'''$. We notice that

$$s_2s_1^4s_2^2s_1^{-2}s_2^2u_1 + s_2s_1^2s_2^2s_1^{-2}s_2^2u_1$$
$$\subset s_2s_1^3(s_1s_2^2s_1^{-1})s_1^{-1}s_2^2u_1 + \omega(s_2s_1^{-2}s_2^{-1})s_2^3u_1$$
$$\subset (s_2s_1^3s_2^{-1})s_1(s_1s_2s_1^{-1})s_2^2u_1 + \omega s_1^{-2}(s_1s_2^{-2}s_1^{-1})s_1^2s_2^3u_1$$
$$\subset s_1^{-1}s_2^3s_1^2s_2^{-1}s_1s_2^3u_1 + \underline{s_1^{-2}\omega s_2^{-1}s_1^{-2}s_2s_1^2s_2^3u_1}$$

Therefore, we have to prove that the element $s_2^3s_1^2s_2^{-1}s_1s_2^3$ is inside U'''. For this purpose, we expand s_2^3 as a linear combination of s_2^2, s_2, 1 s_2^{-1} and s_2^{-2} and we have:

$$s_2^3s_1^2s_2^{-1}s_1s_2^3 \in R_5s_2^3s_1^3(s_1^{-1}s_2^{-1}s_1)s_2^2 + \underline{R_5s_2^3s_1^2(s_2^{-1}s_1s_2)} + \underline{R_5s_2^3s_1^2s_2^{-1}} +$$
$$+ R_5s_2^3s_1^2s_2^{-1}s_1s_2^{-1} + R_5s_1^{-1}(s_1s_2^2s_1^{-1})s_1(s_2s_1^2s_2^{-1})s_1s_2^{-2}$$
$$\in u_2u_1s_2s_1^{-1}s_2 + u_2u_1s_2^{-1}s_1s_2^{-1}u_1 + u_1s_2^{-1}s_1^2s_2^3s_1^2s_2^{-2} + U'''.$$

However, by Proposition 4.4 we have that $u_2u_1s_2s_1^{-1}s_2$ and $u_2u_1s_2^{-1}s_1s_2^{-1}$ are subsets of U'''. Therefore, we only need to prove that the element $s_2^{-1}s_1^2s_2^3s_1^2s_2^{-2}$ is inside U'''. We expand s_2^3 as a linear combination of s_2^2, s_2, 1 s_2^{-1} and s_2^{-2} and we have that $s_2^{-1}s_1^2s_2^3s_1^2s_2^{-2} \in \Phi(s_2V) + \underline{s_2^{-1}(s_1^2s_2s_1)s_1s_2^{-2}} + \Phi(s_2u_1s_2s_1^{-2}s_2^2) + R_5s_2^{-1}s_1^2s_2^{-2}s_1^2s_2^{-2}$. However, by the definition of U''' we have that $s_2^{-1}s_1^2s_2^{-2}s_1^2s_2^{-2} \in U'''$. Moreover, by relation (3) and by the previous case (case C1) we have that $\Phi(s_2V)+\Phi(s_2u_1s_2s_1^{-2}s_2^2) \subset \Phi(U''') \overset{4.1}{\subset} U'''$.

C3. We will prove that $s_2u_1s_2^2s_1^2s_2^2 \subset U'''$. For this purpose, we expand u_1 as $R_5 + R_5s_1 + R_5s_1^{-1} + R_5s_1^2 + R_5s_1^3$ and we have $s_2u_1s_2^2s_1^2s_2^2 \subset \underline{s_2^3s_1^2s_2^2u_1} + \underline{(s_2s_1s_2^2)s_1^2s_2^2u_1} + s_2s_1^{-1}s_2^2s_1^2s_2^2u_1 + s_2s_1^2s_2^2s_1^2s_2^2u_1 + s_2s_1^3s_2^2s_1^2s_2^2$. However, be Lemma 4.5 we have that

$s_2s_1^3s_2^2s_1^2s_2^2 \subset u_1\omega^3 + U'' \subset U'''$. Therefore, it remains to prove that $s_2s_1^{-1}s_2^2s_1^2s_2^2u_1 + s_2s_1^2s_2^2s_1^2s_2^2u_1 \subset U'''$. We have:

$$\begin{aligned}
s_2s_1^{-1}s_2^2s_1^2s_2^2u_1 + s_2s_1^2s_2^2s_1^2s_2^2u_1 &= s_2^2(s_2^{-1}s_1^{-1}s_2)s_2s_1^2s_2^2u_1 + s_1^{-1}s_1\omega^2s_2\\
&= s_2^2s_1(s_2^{-1}s_1^{-1}s_2)s_1^2s_2^2u_1 + s_1^{-1}\omega^2s_1s_2\\
&= s_2^2s_1^2(s_2^{-1}s_1s_2)s_2u_1 + s_1^{-1}s_2s_1^2s_2^2s_1^2(s_2s_1s_2)\\
&\subset u_2u_1s_2s_1^{-1}s_2u_1 + u_1s_2s_1^2s_2^2s_1^3s_2u_1.
\end{aligned}$$

By Lemma 4.4(ii) it will be sufficient to prove that $s_2s_1^2s_2^2s_1^3s_2 \in U'''$. We expand s_2^3 as a linear combination of s_2^2, s_2, 1 s_2^{-1} and s_2^{-2} and we have:

$$\begin{aligned}
s_2s_1^2s_2^2s_1^3s_2 &\in R_5\omega^2 + \underline{R_5s_2s_1^2(s_2^2s_1s_2)} + \underline{R_5s_2s_1^2s_2^2} + R_5s_2s_1(s_1s_2^2s_1^{-1})s_2 +\\
&\quad + R_5s_2s_1^2s_2^2s_1^{-2}s_2\\
&\in \underline{R_5(s_2s_1s_2^{-1})s_1^2s_2^2} + R_5s_1^{-1}(s_1s_2s_1^2)s_2^2s_1^{-2}s_2 + U'''\\
&\in u_1s_2^2(s_1s_2^3s_1^{-1})s_1^{-1}s_2 + U'''\\
&\in u_1u_2u_1s_2s_1^{-1}s_2u_1 + U'''.
\end{aligned}$$

The result follows from Proposition 4.4(ii).

C4. We will prove that $s_2u_1s_2^{-2}s_1^2s_2^{-2}u_1 \subset U'''$. Since $s_2u_1s_2^{-2}s_1^2s_2^{-2}u_1 = (s_2u_1s_2^{-1})s_2^{-1}s_1^2s_2^{-2}u_1 = s_1^{-1}u_2s_1s_2^{-1}s_1^2s_2^{-2}u_1$, it will be sufficient to prove that $u_2s_1s_2^{-1}s_1^2s_2^{-2} \subset U'''$. We expand u_2 as $R_5 + R_5s_2 + R_5s_2^{-1} + R_5s_2^2 + R_5s_2^3$ and we have: $u_2s_1s_2^{-1}s_1^2s_2^{-2} \subset \underline{R_5s_1s_2^{-1}s_1^2s_2^{-2}} + \underline{R_5(s_2s_1s_2^{-1})s_1^2s_2^{-2}} + \Phi(u_1s_2u_1s_2s_1^{-2}s_2^2) + R_5s_2^2s_1s_2^{-1}s_1^2s_2^{-2} + R_5s_2^3s_1s_2^{-1}s_1^2s_2^{-2}$. By the first case (case C1) we have that $\Phi(u_1s_2u_1s_2s_1^{-2}s_2^2) \subset u_1\Phi(U''')u_1 \overset{4.1}{\subset} U'''$. It remains to prove that the elements $s_2^2s_1s_2^{-1}s_1^2s_2^{-2}$ and $s_2^3s_1s_2^{-1}s_1^2s_2^{-2}$ are inside U'''. We have: $s_2^2s_1s_2^{-1}s_1^2s_2^{-2} = s_2(s_2s_1s_2^{-1})s_1^2s_2^{-2} = s_2s_1^{-1}(s_2s_1^3s_2^{-1})s_2^{-1} = s_2s_1^{-2}s_2^2(s_2s_1s_2^{-1}) = s_1^{-1}(s_1s_2s_1^{-1})s_1^{-1}s_2^2s_1^{-1}s_2s_1 = s_1^{-1}s_2^{-1}(s_1s_2s_1^{-1})s_2^2s_1^{-1}s_2s_1 = \underline{s_1^{-1}s_2^{-2}(s_1s_2^3s_1^{-1})s_2s_1}$. By using analogous calculations we have $s_2^3s_1s_2^{-1}s_1^2s_2^{-2} = s_2^2(s_2s_1s_2^{-1})s_1(s_1s_2^{-2}s_1^{-1})s_1 = s_2^2s_1^{-2}(s_1s_2s_1^2)s_2^{-1}s_1^{-2}s_2s_1 \in s_2^2s_1^{-2}s_2^2s_1^{-1}s_2u_1$. We expand s_1^{-2} as a linear combination of s_1^{-1}, 1, s_1, s_2^2 and s_2^3 and we have:

$$\begin{aligned}
s_2^2s_1^{-2}s_2^2s_1^{-1}s_2 &\in R_5s_2^2s_1^{-1}s_2^2s_1^{-1}s_2 + \underline{R_5s_2^4s_1^{-1}s_2} + \underline{R_5s_2(s_2s_1s_2^2)s_1^{-1}s_2} +\\
&\quad + R_5s_2^2s_1^2s_2^2s_1^{-1}s_2 + R_5s_2^2s_1^3s_2^2s_1^{-1}s_2\\
&\in R_5s_2^3(s_2^{-1}s_1^{-1}s_2)s_2s_1^{-1}s_2 + R_5s_2^2s_1(s_1s_2^2s_1^{-1})s_2 +\\
&\quad + R_5s_2^2s_1^2(s_1s_2^2s_1^{-1})s_2 + U'''\\
&\in R_5s_2^3s_1(s_2^{-1}s_1^{-1}s_2)s_1^{-1}s_2 + R_5s_2(s_2s_1s_2^{-1})s_1^2s_2^2 +\\
&\quad + R_5s_2(s_2s_1^2s_2^{-1})s_1^2s_2^2 + U'''\\
&\in \underline{R_5s_2^3s_1^2(s_2^{-1}s_1^{-2}s_2)} + R_5s_2s_1^{-1}s_2s_1^3s_2^2 + R_5s_2s_1^{-1}s_2^2s_1^3s_2^2 + U'''
\end{aligned}$$

Therefore, it remains to prove that $B := R_5 s_2 s_1^{-1} s_2 s_1^3 s_2^2 + R_5 s_2 s_1^{-1} s_2^2 s_1^3 s_2^2 \subset U'''$. We expand s_1^3 as a linear combination of s_1^2, s_1, 1 s_1^{-1} and s_1^{-2} and we have that $B \subset R_5 s_2 s_1^{-1} s_2 (R_5 s_1^2 + R_5 s_1 + R_5 + R_5 s_1^{-1} + R_5 s_1^{-2}) s_2^2 + R_5 s_2 s_1^{-1} s_2^2 (R_5 s_1^2 + R_5 s_1 + R_5 + R_5 s_1^{-1} + R_5 s_1^{-2}) s_2^2$. By cases C1, C2 and C3 we have:

$$\begin{aligned} B &\subset R_5 s_2 s_1^{-1} \omega s_2 + \underline{R_5 s_2 s_1^{-1}(s_2 s_1 s_2^2)} + \underline{R_5 s_2 s_1^{-1} s_2^3 u_1} + R_5 s_2 s_1^{-1} s_2 s_1^{-1} s_2^2 + \\ &\quad + \underline{R_5 s_2 s_1^{-1}(s_2^2 s_1 s_2) s_2} + \underline{R_5 s_2 s_1^{-1} s_2^4} + R_5 s_2 s_1^{-1} s_2^2 s_1^{-1} s_2^2 + U''' \\ &\subset R_5 s_2 \omega s_1^{-1} s_2 + R_5 s_2 s_1^{-1} s_2 s_1^{-1} s_2^2 + R_5 s_2 s_1^{-1} s_2^2 s_1^{-1} s_2^2 + U''' \\ &\subset u_2 u_1 s_2 s_1^{-1} s_2 + R_5 s_2^2 (s_2^{-1} s_1^{-1} s_2) s_1^{-1} s_2^2 + R_5 s_2 s_1^{-2} (s_1 s_2^2 s_1^{-1}) s_2^2 + U''' \\ &\overset{4.4}{\subset} R_5 s_2^2 s_1 (s_2^{-1} s_1^{-2} s_2) s_2 + R_5 (s_2 s_1^{-2} s_2^{-1}) s_1^2 s_2^3 + U''' \\ &\subset R_5 s_2^2 s_1^2 s_2^{-1} (s_2^{-1} s_1^{-1} s_2) + U''' \\ &\subset u_1 u_2 u_1 s_2^{-1} s_1 s_2^{-1} + U'''. \end{aligned}$$

The result follows from Proposition 4.4(ii).

C5. We will prove that $s_2 u_1 s_2^2 s_1^2 s_2^{-2} u_1 \subset U'''$. For this purpose, we use straightforward calculations and we have $s_2 u_1 s_2^2 s_1^2 s_2^{-2} = (s_2 u_1 s_2^{-1}) s_2^2 (s_2 s_1^2 s_2^{-1}) s_2^{-1} = s_1^{-1} u_2 (s_1 s_2^2 s_1^{-1}) s_2 (s_2 s_1 s_2^{-1}) = s_1^{-1} u_2 s_1 (s_1 s_2^2 s_1^{-1}) s_2 s_1 = s_1^{-1} u_2 (s_2 s_1 s_2^{-1}) s_1^2 s_2^2 s_1 = s_1^{-2} (s_1 u_2 s_1^{-1}) s_2 s_1^3 s_2^2 s_1 = s_1^{-2} s_2^{-1} u_1 s_2^2 s_1^3 s_2^2 s_1$, meaning that $s_2 u_1 s_2^2 s_1^2 s_2^{-2} u_1 \subset u_1 s_2^{-1} u_1 s_2^2 s_1^3 s_2^2 u_1$. Hence, we have to prove that $s_2^{-1} u_1 s_2^2 s_1^3 s_2^2 \subset U'''$. For this purpose, we expand s_1^3 as a linear combination of s_1^2, s_1, 1 s_1^{-1} and s_1^{-2} and we have that $s_2^{-1} u_1 s_2^2 s_1^3 s_2^2 \subset \Phi(s_2 V + s_2 u_1 s_2^{-2} s_1^2 s_2^{-2}) + s_2^{-1} u_1 s_2^2 s_1 s_2^2 + s_2^{-1} u_1 s_2^2 s_1^{-1} s_2^2$. By relation (3) and case C4 we have that $\Phi(s_2 V + s_2 u_1 s_2^{-2} s_1^2 s_2^{-2}) \subset \Phi(U''') \overset{4.1}{\subset} U'''$. Moreover, $s_2^{-1} u_1 s_2^2 s_1 s_2^2 = s_2^{-1} u_1 (s_2^2 s_1 s_2) s_2 = s_2^{-1} u_1 \omega = \underline{\underline{s_2^{-1} \omega u_1}}$. It remains to prove that $s_2^{-1} u_1 s_2^2 s_1^{-1} s_2^2 \subset U'''$. We have: $s_2^{-1} u_1 s_2^2 s_1^{-1} s_2^2 = (s_2^{-1} u_1 s_2) s_2 s_1^{-1} s_2^2 = s_1 u_2 (s_1^{-1} s_2 s_1) s_1^{-2} s_2^2 = s_1 u_2 s_1 (s_2^{-1} s_1^{-2} s_2) s_2 \subset u_1 u_2 u_1 s_2^{-1} s_1 s_2^{-1} u_1$. The result follows from Proposition 4.4(i). □

From now on we will double-underline the elements of the form $u_1 s_2^{\pm} u_1 u_2 u_1 u_2 u_1$ and we will use the fact that they are elements of U''' (Proposition 4.6) without mentioning it.

We can now prove the following lemma that helps us to "replace" inside the definition of U'''' the element ω^4 by the element $s_2^{-2} s_1^2 s_2^2 s_1^3 s_2^2$ modulo U'''.

Lemma 4.7. $s_2^{-2} s_1^2 s_2^2 s_1^3 s_2^2 \in u_1 \omega^3 + u_1^{\times} \omega^4 + u_1 s_2 u_1 u_2 u_1 u_2 u_1 \subset U''''$.

Proof. In this proof we will double-underline only the elements of the form $u_1 s_2 u_1 u_2 u_1 u_2 u_1$ (and not of the form $u_1 s_2^{-1} u_1 u_2 u_1 u_2 u_1$). The fact that $u_1 \omega^3 + u_1^{\times} \omega^4 + u_1 s_2 u_1 u_2 u_1 u_2 u_1$ is a subset of U'''' follows from the definition of U'''' and Proposition 4.6. As a result, we restrict ourselves to proving that $s_2^{-2} s_1^2 s_2^2 s_1^3 s_2^2 \in u_1 \omega^3 + u_1^{\times} \omega^4 + u_1 s_2 u_1 u_2 u_1 u_2 u_1$. We first notice that

$$\begin{aligned} s_2^{-2}s_1^2s_2^2s_1^3s_2^2 &= s_1(s_1^{-1}s_2^{-2}s_1)s_2^{-2}(s_2^2s_1s_2)s_2s_1^2(s_1s_2^2s_1^{-1})s_1^{-1}s_1^2 \\ &= s_1s_2s_1^{-2}s_2^{-3}s_1\omega s_1^2s_2^{-1}s_1(s_1s_2s_1^{-1})s_1^2 \\ &= s_1s_2s_1^{-2}s_2^{-3}s_1^3(s_2s_1^3s_2^{-1})s_1s_2s_1^2 \\ &= s_1s_2s_1^{-2}s_2^{-3}s_1^2s_2^3s_1^2s_2s_1^2 \\ &\in u_1s_2s_1^{-2}s_2^{-3}s_1^2s_2^3s_1^2s_2u_1. \end{aligned}$$

We expand s_2^{-3} as a linear combination of s_2^{-2}, s_2^{-1}, 1, s_2 and s_2^2, where the coefficient of s_2^2 is invertible, and we have:

$$\begin{aligned} s_2s_1^{-2}s_2^{-3}s_1^2s_2^3s_1^2s_2 \in R_5s_2s_1^{-2}s_2^{-2}s_1^2s_2^3s_1^2s_2 + R_5s_2s_1^{-2}s_2^{-1}s_1^2s_2^3s_1^2s_2 + \underline{\underline{R_5s_2s_2^3s_1^2s_2}} + \\ + R_5s_2s_1^{-2}s_2s_1^2s_2^3s_1^2s_2 + u_1^\times s_2s_1^{-2}s_2^2s_1^2s_2^3s_1^2s_2u_1^\times. \end{aligned}$$

However, we notice that $s_2s_1^{-2}s_2^{-2}s_1^2s_2^3s_1^2s_2 = s_2s_1^{-1}(s_1^{-1}s_2^{-2}s_1)s_1s_2^3s_1^2s_2 = s_2s_1^{-1}s_2^2\omega^{-1}s_1s_2^3s_1^2s_2 = s_2s_1^{-1}s_2^2s_1\omega^{-1}s_2^3s_1^2s_2 = s_2s_1^{-1}s_2^2s_1(s_2^{-1}s_1^{-2}s_2)\omega = s_2s_1^{-1}s_2^2s_1^2s_2^{-2}\omega s_1^{-1} = \underline{\underline{s_2s_1^{-1}s_2^2s_1^2(s_2^{-1}s_1^2s_2)}}s_1^{-1}$. Moreover, we have $s_2s_1^{-2}s_2^{-1}s_1^2s_2^3s_1^2s_2 = s_2s_1^{-2}(s_2^{-1}s_1^2s_2)s_2^2s_1^2s_2 = s_2s_1^{-1}s_2^2(s_1^{-1}s_2^2s_1)s_2(s_2^{-1}s_1s_2) = \underline{\underline{s_2s_1^{-1}s_2^3(s_1^3s_2s_1)s_1^{-2}}}$. We also have $s_2s_1^{-2}s_2s_1^2s_2^3s_1^2s_2 = s_2s_1^{-3}(s_1s_2s_1^2)s_2^3s_1^2s_2 = s_2s_1^{-3}s_2(s_2s_1s_2^4)s_1^2s_2 \in s_2s_1^{-3}(s_2u_1s_2u_1s_2u_1)$. However, by Lemma 4.3(i) we have that $s_2s_1^{-3}(s_2u_1s_2u_1s_2u_1) \subset s_2s_1^{-3}(\omega^2u_1 + u_1u_2u_1u_2u_1) \subset s_2\omega^2u_1 + \underline{\underline{u_1s_2u_1u_2u_1u_2u_1}}$. By Lemma 4.3(ii) we also have $s_2\omega^2u_1 \subset \underline{\underline{u_1s_2u_1u_2u_1u_2u_1}}$.

It remains to prove that $s_2s_1^{-2}s_2^2s_1^2s_2^3s_1^2s_2 \in u_1\omega^3 + u_1^\times\omega^4 + u_1s_2u_1u_2u_1u_2u_1$. We have:

$$\begin{aligned} s_2s_1^{-2}s_2^2s_1^2s_2^3s_1^2s_2 &= s_2(-de^{-1}s_1^{-1} - ce^{-1} - e^{-1}bs_1 - e^{-1}as_1^2 + e^{-1}s_1^3)s_2^2s_1^2s_2^3s_1^2s_2 \\ &\in R_5s_2s_1^{-1}s_2^2s_1^2s_2^3s_1^2s_2 + R_5s_2^3s_1^2s_2^3s_1^2s_2 + \underline{\underline{R_5(s_2s_1s_2^2)s_1^2s_2^3s_1^2s_2u_1}} + \\ &\quad + R_5s_2s_1^2s_2^2s_1^2s_2^3s_1^2s_2 + u_1^\times s_2s_1^3s_2^2s_1^2s_2^2\omega. \end{aligned}$$

We first notice that we have $s_2s_1^{-1}s_2^2s_1^2s_2^3s_1^2s_2 = s_2(s_1^{-1}s_2^2s_1)s_1s_2^3s_1^2s_2 = s_2^2s_1^2(s_2^{-1}s_1s_2)s_2^2s_1^2s_2 = s_1(s_1^{-1}s_2^2s_1)s_1^2s_2(s_1^{-1}s_2^2s_1)(s_1s_2s_1)s_1^{-1} = s_1s_2s_1^2(s_2^{-1}s_1^2s_2)s_2s_1^3s_2s_1^{-1} = \underline{\underline{s_1s_2s_1^3s_2^3(s_2^{-1}s_1^{-1}s_2)(s_1^3s_2s_1)s_1^{-2}}}$. Moreover, we have that $s_2^3s_1^2s_2^3s_1^2s_2 = s_1(s_1^{-1}s_2^3s_1)s_1s_2^3s_1^2s_2 = s_1s_2s_1^3(s_2^{-1}s_1s_2)s_2^2s_1(s_1s_2s_1)s_1^{-1} = \underline{\underline{s_1s_2s_1^4s_2(s_1^{-1}s_2^2s_1)s_2s_1s_2s_1^{-1}}}$. Using analogous calculations, we also have that $s_2s_1^2s_2^2s_1^2s_2^3s_1^2s_2 = s_1^{-1}(s_1s_2s_1^2)s_2^2s_1^2s_2^3s_1^2s_2 = s_1^{-1}s_2(s_2s_1s_2^3)s_1^2s_2^3s_1^2s_2 = s_1^{-1}s_2s_1^2(s_1s_2s_1^3)s_2^3s_1^2s_2 = s_1^{-1}s_2s_1^2s_2^2(s_2s_1s_2^4)s_1^2s_2 \in u_1\omega(s_2u_1s_2u_1s_2u_1)$. However, by Lemma 4.3(i) we have that $u_1\omega(s_2u_1s_2u_1s_2u_1) \subset u_1\omega(\omega^2u_1 + u_1u_2u_1u_2u_1) \subset u_1\omega^3 + \underline{\underline{u_1\omega u_2u_1u_2u_1}}$.

In order to finish the proof, it remains to prove that $s_2s_1^3s_2^2s_1^2s_2^2\omega \in u_1^\times\omega^4 + u_1s_2u_1u_2u_1u_2u_1$. We use Lemma 4.5 and we have:

$$\begin{aligned}
s_2s_1^3s_2^2s_1^2s_2^2\omega &\in u_1^{\times}(u_1s_2u_1s_2s_1^3s_2+u_1s_2^2s_1^3s_2s_1^{-1}s_2+u_1u_2u_1u_2u_1+u_1^{\times}\omega^3)\omega\\
&\in u_1s_2u_1s_2s_1^4(s_1^{-1}s_2^2s_1)s_1s_2+u_1s_2^2s_1^3s_2(s_1^{-1}s_2s_1)s_1^{-1}\omega+u_1^{\times}\omega^4\\
&\in u_1s_2u_1s_2s_1^4s_2s_1^2(s_2^{-1}s_1s_2)+u_1s_2^2s_1^3s_2^2s_1s_2^{-1}\omega+u_1^{\times}\omega^4\\
&\in u_1s_2u_1s_2s_1^4s_2s_1^3s_2u_1+u_1s_2^2s_1^3s_2^2s_1^3s_2+u_1^{\times}\omega^4\\
&\in u_1s_2u_1(s_2u_1s_2u_1s_2u_1)+u_1(s_1^{-1}s_2^2s_1)s_1^2s_2^2s_1^3s_2+u_1^{\times}\omega^4\\
&\overset{4.3(\mathrm{i})}{\in} u_1s_2u_1(\omega^2u_1+u_1u_2u_1u_2u_1)+u_1s_2s_1^2(s_2^{-1}s_1^2s_2)s_2s_1^3s_2+u_1^{\times}\omega^4\\
&\in u_1s_2\omega^2u_1+u_1s_2u_1u_2u_1u_2u_1+u_1s_2s_1^3s_2^2(s_1^{-1}s_2s_1)s_1^2s_2+u_1^{\times}\omega^4\\
&\overset{4.3(\mathrm{ii})}{\in} u_1s_2u_1u_2u_1u_2u_1+\underline{\underline{u_1s_2s_1^3s_2^3s_1(s_2^{-1}s_1^2s_2)}}+u_1^{\times}\omega^4. \qquad \square
\end{aligned}$$

Proposition 4.8.

(i) $s_2u_1u_2u_1s_2s_1^{-1}s_2\subset U''''$.
(ii) $s_2u_1u_2u_1s_2^{-1}s_1s_2^{-1}\subset U''''$.
(iii) $s_2u_1u_2u_1u_2\omega\subset U''''$.

Proof.

(i) By Proposition 4.4(ii) we have $s_2u_1u_2u_1s_2s_1^{-1}s_2\subset s_2u_1(u_1\omega^2+R_5s_2^2s_1^{-2}s_2s_1^{-1}s_2+u_1u_2u_1u_2u_1)$ and, hence, by Lemma 4.3(ii) we have $s_2u_1u_2u_1s_2s_1^{-1}s_2\subset s_2u_1s_2^2s_1^{-2}s_2s_1^{-1}s_2+\underline{\underline{s_2u_1u_2u_1u_2u_1}}+U''''$. As a result, we must prove that $s_2u_1s_2^2s_1^{-2}s_2s_1^{-1}s_2\subset U''''$. For this purpose, we expand u_1 as $R_5+R_5s_1+R_5s_1^{-1}+R_5s_1^2+R_5s_1^3$ and we have:

$$\begin{aligned}
s_2u_1s_2^2s_1^{-2}s_2s_1^{-1}s_2\subset u_2u_1s_2s_1^{-1}s_2&+\underline{\underline{R_5(s_2s_1s_2^2)s_1^{-2}s_2s_1^{-1}s_2}}+\\
&+R_5s_2s_1^{-1}s_2^2s_1^{-2}s_2s_1^{-1}s_2+R_5s_2s_1^2s_2^2s_1^{-2}s_2s_1^{-1}s_2+\\
&+R_5s_2s_1^3s_2^2s_1^{-2}s_2s_1^{-1}s_2.
\end{aligned}$$

By Proposition 4.4(ii) we have that $u_2u_1s_2s_1^{-1}s_2\subset U''''$. Moreover, $s_2s_1^2s_2^2s_1^{-2}s_2s_1^{-1}s_2=s_1^{-1}(s_1s_2s_1^2)s_2^2s_1^{-3}(s_1s_2s_1^{-1})s_2=s_1^{-1}s_2(s_2s_1s_2^3)s_1^{-3}s_2^{-1}s_1s_2^2=\underline{\underline{s_1^{-1}s_2s_1^3(s_2s_1^{-2}s_2^{-1})s_1s_2^2}}$. We also notice that $s_2s_1^3s_2^2s_1^{-2}s_2s_1^{-1}s_2=s_1^{-1}(s_1s_2s_1^3)s_2^2s_1^{-3}(s_1s_2s_1^{-1})s_2=s_1^{-1}s_2^3(s_1s_2^3s_1^{-1})s_1^{-2}s_2^{-1}s_1s_2^2=s_1^{-1}s_2^3s_1^3(s_2s_1^{-2}s_2^{-1})s_1s_2^2=s_1^{-2}(s_1s_2s_1^{-1})s_1(s_2s_1^2s_2^{-1})s_2^{-1}s_1^2s_2^2=s_1^{-2}s_2^{-1}(s_1s_2^3s_1^{-1})s_2^{-1}(s_2s_1^2s_2^{-1})s_1^2s_2^2\in u_1s_2^{-2}s_1^2s_2^2s_1^3s_2^2\overset{4.7}{\subset}U''''$.
It remains to prove that the element $s_2s_1^{-1}s_2^2s_1^{-2}s_2s_1^{-1}s_2$ is inside U''''. We expand s_2^2 as a linear combination of s_2, 1, s_2^{-1}, s_2^{-2} and s_2^{-3} and we have

$$s_2s_1^{-1}s_2^2s_1^{-2}s_2s_1^{-1}s_2\in s_2s_1^{-1}(R_5s_2+R_5+R_5s_2^{-1}+R_5s_2^{-2}+R_5s_2^{-3})s_1^{-2}s_2s_1^{-1}s_2$$

$$\begin{aligned}
&\in R_5s_2s_1^{-2}(s_1s_2s_1^{-1})s_1^{-1}(s_2s_1^{-1}s_2^{-1})s_2^2+\underline{\underline{R_5s_2s_1^{-3}s_2s_1^{-1}s_2}}+\\
&\quad+\underline{\underline{R_5(s_2s_1^{-1}s_2^{-1})s_1^{-2}s_2s_1^{-1}s_2}}+\\
&\quad+\underline{\underline{R_5(s_2s_1^{-1}s_2^{-1})(s_2^{-1}s_1^{-2}s_2)s_1^{-1}s_2}}+\\
&\quad+R_5(s_2s_1^{-1}s_2^{-1})s_2^{-1}(s_2^{-1}s_1^{-2}s_2)s_1^{-1}s_2\\
&\in \underline{\underline{R_5s_2s_1^{-2}s_2^{-1}s_1(s_2s_1^{-2}s_2^{-1})s_1s_2^2}}+\\
&\quad+R_5s_1^{-1}s_2^{-1}s_1^2(s_1^{-1}s_2^{-1}s_1)s_2^{-2}s_1^{-2}s_2+U''''\\
&\in R_5s_1^{-1}s_2^{-1}s_1^2s_2(s_1^{-1}s_2^{-3}s_1)s_1^{-3}s_2+U''''\\
&\in \underline{\underline{R_5s_1^{-1}s_2^{-1}s_1^2s_2^2s_1^{-3}(s_2^{-1}s_1^{-3}s_2)}}+U''''.
\end{aligned}$$

(ii) By Proposition 4.4 (i) we have that $s_2u_1(u_2u_1s_2^{-1}s_1s_2^{-1})\subset s_2u_1(u_1\omega^{-2}+R_5s_2^{-2}s_1^2s_2^{-1}s_1s_2^{-1}+u_1u_2u_1u_2u_1)\subset \underline{\underline{s_2\omega^{-2}u_1}}+s_2u_1s_2^2s_1^{-2}s_2s_1^{-1}s_2+\underline{\underline{s_2u_1u_2u_1u_2u_1}}$. Therefore, it remains to prove that $s_2u_1s_2^2s_1^{-2}s_2s_1^{-1}s_2\subset U''''$. We expand u_1 as $R_5+R_5s_1+R_5s_1^{-1}+R_5s_1^2+R_5s_1^3$ and we have:

$$\begin{aligned}
s_2u_1s_2^2s_1^{-2}s_2s_1^{-1}s_2 &\subset \underline{\underline{R_5s_2^{-1}s_1^2s_2^{-1}s_1s_2^{-1}}}+\underline{\underline{R_5s_1^{-2}(s_1^2s_2s_1)s_2^{-2}s_1^2s_2^{-1}s_1s_2^{-1}}}+\\
&\quad+R_5(s_2s_1^{-1}s_2^{-1})(s_2^{-1}s_1^2s_2)s_2^{-2}s_1s_2^{-1}+\\
&\quad+R_5(s_2s_1^2s_2^{-1})s_2^{-2}(s_2s_1^2s_2^{-1})s_1s_2^{-1}+\\
&\quad+R_5(s_2s_1^3s_2^{-1})s_2^{-1}s_1^2s_2^{-1}(s_1s_2s_1^{-1})s_1\\
&\subset \underline{\underline{R_5s_1^{-1}s_2^{-1}s_1^2s_2^2(s_1^{-1}s_2^{-2}s_1)s_2^{-1}}}+\\
&\quad+\underline{\underline{R_5s_1^{-1}s_2^2(s_1s_2^{-2}s_1^{-1})s_2(s_2s_1s_2^{-1})}}+\\
&\quad+R_5s_1^{-1}s_2^2(s_2s_1s_2^{-1})s_1(s_1s_2^{-2}s_1^{-1})s_2s_1+U''''\\
&\subset R_5s_1^{-1}s_2^2s_1^{-1}(s_2s_1^2s_2^{-1})s_1^{-2}s_2^2s_1+U''''\\
&\subset R_5s_1^{-1}s_2^2s_1^{-3}(s_1s_2^2s_1^{-1})s_2^2s_1+U''''\\
&\subset \underline{\underline{R_5s_1^{-1}s_2(s_2s_1^{-3}s_2^{-1})s_1^2s_2^3s_1}}+U''''.
\end{aligned}$$

(iii) We notice that $s_2u_1u_2u_1u_2\omega=s_2u_1u_2u_1u_2s_1^2s_2s_2u_1u_2u_1u_2(s_2^{-1}s_1^2s_2)\subset s_2u_1u_2u_1u_2s_1s_2^2u_1$. We expand $\mathbf{u_2}$ as $R_5+R_5s_2+R_5s_2^{-1}+R_5s_2^2+R_5s_2^{-2}$ and we have:

$$\begin{aligned}
s_2u_1u_2u_1\mathbf{u_2}s_1s_2^2u_1 &\subset \underline{\underline{s_2u_1u_2u_1s_2^2u_1}}+\underline{\underline{s_2u_1u_2u_1(s_2s_1s_2^2)u_1}}+\\
&\quad+s_2u_1u_2u_1(s_2^{-1}s_1s_2)s_2u_1+s_2u_1u_2u_1s_2(s_2s_1s_2^2)u_1+\\
&\quad+s_2u_1u_2u_1s_2^{-1}(s_2^{-1}s_1s_2)s_2u_1\\
&\subset s_2u_1(u_2u_1s_2s_1^{-1}s_2u_1)+\underline{\underline{s_2u_1u_2u_1\omega u_1}}+
\end{aligned}$$

$$+ s_2u_1u_2u_1(s_2^{-1}s_1s_2)s_1^{-1}s_2u_1 + U''''$$
$$\overset{\text{(i)}}{\subset} s_2u_1u_2u_1s_2s_1^{-2}s_2u_1 + U''''.$$

It remains to prove that $s_2u_1u_2u_1s_2s_1^{-2}s_2u_1 \subset U''''$. For this purpose, we expand s_1^{-2} as a linear combination of s_1^{-1}, 1, s_1, s_1^2, s_1^3 and we have: $s_2u_1u_2u_1s_2s_1^{-2}s_2u_1 \subset s_2u_1u_2u_1s_2s_1^{-1}s_2u_1 + \underline{\underline{s_2u_1u_2u_1s_2^2u_1}} + \underline{\underline{s_2u_1u_2u_1(s_2s_1s_2)u_1}} + s_2u_1u_2u_1\omega u_1+s_2u_1u_2u_1(s_1^{-1}s_2s_1)s_1^2s_2u_1 \overset{(i)}{\subset} \underline{\underline{s_2u_1u_2\omega u_1}}+s_2u_1u_2u_1s_2s_1(s_2^{-1}s_1^2s_2)u_1+ U'''' \subset s_2u_1u_2u_1\omega s_2u_1 + U'''' \subset s_2u_1u_2\omega u_1s_2u_1 + U''''$.
However,

$$\begin{aligned} s_2u_1u_2\omega u_1s_2u_1 &\subset s_2u_1u_2s_1^2s_2(R_5 + R_5s_1 + R_5s_1^{-1} + R_5s_1^2 + R_5s_1^3)s_2u_1 \\ &\subset \underline{\underline{s_2u_1u_2s_1^2s_2^2u_1}} + \underline{\underline{s_2u_1u_2(s_1^2s_2s_1)s_2u_1}} + s_2u_1(u_2u_1s_2s_1^{-1}s_2u_1) + \\ &\quad +s_2u_1u_2s_1^2\omega u_1 + s_2u_1(s_1^{-1}u_2s_1)(s_1s_2s_1^3)s_2u_1 \\ &\overset{\text{(i)}}{\subset} \underline{\underline{s_2u_1u_2\omega u_1}} + s_2u_1s_2u_1(s_2^2s_1s_2)s_2u_1 + U'''' \\ &\subset s_2u_1s_2u_1\omega u_1 + U''''. \end{aligned}$$

The result follows from the fact that $s_2u_1s_2u_1\omega u_1 = \underline{\underline{s_2u_1s_2\omega u_1}}$. □

We can now prove the following lemma that helps us to "replace" inside the definition of U the elements ω^5 and ω^{-5} by the elements $s_2^{-2}s_1^2s_2^3s_1^2s_2^3$ and $s_2^{-2}s_1^2s_2^{-2}s_1^2s_2^{-2}$ modulo U'''', respectively.

Lemma 4.9.

(i) $s_2^{-2}s_1^2s_2^3s_1^2s_2^3 \in u_1^{\times}\omega^5 + U''''$.
(ii) $s_2^{-2}s_1^2s_2^{-2}s_1^2s_2^{-2} \in u_1^{\times}\omega^{-5} + U''''$.
(iii) $s_2^{-2}u_1s_2^{-2}s_1^2s_2^{-2} \subset U$.

Proof.

(i) We notice that $s_2^{-2}s_1^2s_2^3s_1^2s_2^3 = s_2^{-2}s_1^2s_2^3s_1(s_1s_2^3s_1^{-1})s_1 = s_2^{-2}s_1^2s_2^2(s_2s_1s_2^{-1}) \times s_1^2(s_1s_2s_1^{-1})s_1^2 = s_2^{-2}s_1^2s_2^2s_1^{-1}(s_2s_1^3s_2^{-1})s_1s_2s_1^2 = s_2^{-2}s_1^2s_2^2s_1^{-2}s_2^2\omega s_1^2$. We expand s_1^{-2} as a linear combination of s_1^{-1}, 1, s_1, s_1^2 and s_1^3, where the coefficient of s_1^3 is invertible, and we have:

$$\begin{aligned} s_2^{-2}s_1^2s_2^2s_1^{-2}s_2^2\omega s_1^2 \in\ & s_2^{-2}s_1(s_1s_2^2s_1^{-1})s_2^2\omega u_1 + s_2^{-1}(s_2^{-1}s_1^4s_2)s_2^2\omega u_1 + \\ & + s_2^{-2}s_1^2(s_2^2s_1s_2)s_2\omega u_1 + s_2^{-1}(s_2^{-1}s_1^2s_2)s_2s_1^2s_2^2\omega s_2\omega u_1 + \\ & + u_1^{\times}s_2^{-2}s_1^2s_2^2s_1^3s_2^2\omega s_1^2 \end{aligned}$$

$$\overset{4.7}{\in} \underbrace{\underline{s_1(s_1^{-1}s_2^{-2}s_1)s_2^{-1}s_1^2s_2^3\omega u_1 + s_1(s_1^{-1}s_2^{-1}s_1)s_2^4s_1^{-1}s_2^2s_1\omega u_1}}_{\in u_1s_2u_1u_2u_1u_2\omega u_1} +$$
$$+ s_2^{-2}s_1^3\omega^2u_1 + (s_2^{-1}s_1s_2)s_2(s_1^{-1}s_2s_1)s_1s_2^2\omega u_1 +$$
$$+ u_1^{\times}(u_1\omega^3 + u_1^{\times}\omega^4 + u_1s_2u_1u_2u_1u_2u_1)\omega s_1^2$$
$$\in \underline{\underline{s_2^{-2}\omega^2u_1}} + s_1s_2s_1^{-1}s_2^2s_1(s_2^{-1}s_1s_2)s_2\omega u_1 + u_1\omega^3 + u_1^{\times}\omega^5 +$$
$$+ u_1s_2u_1u_2u_1u_2u_1\omega u_1$$
$$\in s_1s_2s_1^{-1}(u_2s_1^2s_2s_1^{-1}s_2)\omega u_1 + u_1^{\times}\omega^5 + u_1s_2u_1u_2u_1u_2u_1\omega u_1 +$$
$$+ U''''.$$

By Proposition 4.4(ii) we have that $s_2s_1^{-1}(u_2s_1^2s_2s_1^{-1}s_2)\omega u_1 \subset s_2s_1^{-1}(u_1\omega^2 + R_5s_2^2s_1^{-2}s_2s_1^{-1}s_2 + u_1u_2u_1u_2u_1)\omega u_1 \overset{4.7}{\in} s_2\omega^2u_1 + u_1s_2u_1u_2u_1u_2u_1\omega u_1$ and, hence, the element $s_2^{-2}s_1^2s_2^2s_1^{-2}s_2^2\omega s_1^2$ is inside $s_2\omega^2u_1 + u_1^{\times}\omega^5 + u_1s_2u_1u_2u_1u_2u_1\omega u_1 + U''''$. We notice that $u_1s_2u_1u_2u_1u_2u_1\omega u_1 = u_1s_2u_1u_2u_1u_2\omega u_1$ and, hence, by Lemma 4.3(ii) and Proposition 4.8(iii) we have that the element $s_2^{-2}s_1^2s_2^2s_1^{-2}s_2^2\omega s_1^2$ is inside $u_1^{\times}\omega^5 + U''''$.

(ii)

$$s_2^{-2}s_1^2s_2^{-2}s_1^2s_2^{-2} = s_2^{-2}(as_1 + b + cs_1^{-1} + ds_1^{-2} + es_1^{-3})s_2^{-2}s_1^2s_2^{-2}$$
$$\in \underline{\underline{s_1(s_1^{-1}s_2^{-2}s_1)s_2^{-2}s_1^2s_2^{-2}}} + \underline{R_5s_2^{-4}s_1^2s_2} + \underline{\underline{R_5s_2^{-1}(s_2^{-1}s_1^{-1}s_2^{-2})s_1^2s_2^{-2}}} +$$
$$+ R_5s_2^{-1}(s_2^{-1}s_1^{-2}s_2)s_2^{-3}s_1^2s_2^{-2} + R_5s_2^{-2}s_1^{-2}(s_1^{-1}s_2^{-2}s_1)s_1s_2^{-2}$$
$$\in R_5s_2^{-1}s_1s_2^{-2}(s_1^{-1}s_2^{-3}s_1)s_1s_2^{-2} +$$
$$+ R_5s_2^{-1}(s_2^{-1}s_1^{-2}s_2)s_1^{-2}(s_2^{-1}s_1s_2)s_2^{-3} + U''''$$
$$\in R_5s_2^{-1}s_1^2(s_1^{-1}s_2^{-1}s_1^{-3})s_2^{-1}s_1s_2^{-2} +$$
$$+ R_5s_2^{-1}s_1s_2^{-1}(s_2^{-1}s_1^{-2}s_2)s_1^{-1}s_2^{-3} + U''''$$
$$\in \underline{\underline{R_5s_2^{-1}s_1^2s_2^{-2}(s_2^{-1}s_1^{-1}s_2^{-2})s_1s_2^{-2}}} +$$
$$+ R_5s_2^{-1}s_1(s_2^{-1}s_1s_2)s_2^{-2}s_1^{-2}s_2^{-3} + U''''$$
$$\in R_5(s_2^{-1}s_1^2s_2)s_1^{-1}s_2^{-2}s_1^{-2}s_2^{-3} + U''''$$
$$\in \Phi(u_1s_2^{-2}s_1^2s_2^3s_1^2s_2^3) + U''''.$$

The result then follows from (i) and Lemma 4.1.

(iii) We expand u_1 as $R_5 + R_5s_1 + R_5s_1^{-1} + R_5s_1^2 + R_5s_1^{-2}$ and we have that

$$s_2^{-2}u_1s_2^{-2}s_1^2s_2^{-2} \subset \underline{R_5s_2^{-4}s_1^2s_2^{-2}} + \underline{\underline{R_5s_2^{-1}(s_2^{-1}s_1s_2)s_2^{-3}s_1^2s_2^{-2}}} +$$
$$+ \underline{\underline{R_5(s_2^{-2}s_1^{-1}s_2^{-1})s_2^{-1}s_1^2s_2^{-2}}} + R_5s_2^{-2}s_1^2s_2^{-2}s_1^2s_2^{-2} +$$
$$+ R_5s_2^{-2}s_1^{-2}s_2^{-2}s_1^2s_2^{-2}.$$

The element $s_2^{-2}s_1^2s_2^{-2}s_1^2s_2^{-2}$ is inside U'''', by (ii). Moreover, we have $s_2^{-2}s_1^{-2}s_2^{-2}s_1^2s_2^{-2} = s_2^{-1}\omega^{-1}(s_2^{-1}s_1^2s_2)s_2^{-3} = s_2^{-1}\omega^{-1}s_1s_2^2s_1^{-1}s_2^{-3} = s_2^{-1}s_1\omega^{-1}s_2^2s_1^{-1}s_2^{-3} = \underline{\underline{s_2^{-1}s_1^2(s_1^{-1}s_2^{-1}s_1^{-2})s_2s_1^{-1}s_2^{-3}}}$. □

We can now prove the main theorem of this section.

Theorem 4.10.

(i) $U = U'''' + u_1\omega^{-5}$.
(ii) $H_5 = U$.

Proof.

(i) By definition, $U = U'''' + u_1\omega^5 + u_1\omega^{-5}$. Hence, it is enough to prove that $\omega^{-5} \in u_1^\times\omega^5 + U''''$. By Lemma 4.9(ii) we have that $\omega^{-5} \in u_1^\times s_2^{-2}s_1^2s_2^{-2}s_1^2s_2^{-2} + U''''$. We expand $\mathbf{s_2^{-2}}$ as a linear combination of s_2^{-1}, 1, s_2, s_2^2 and s_2^3, where the coefficient of s_2^3 is invertible, and we have:

$$\begin{aligned}
s_2^{-2}s_1^2s_2^{-2}s_1^2\mathbf{s_2^{-2}} &\in R_5s_2^{-1}(s_2^{-1}s_1^2s_2)s_2^{-3}s_1^2s_2^{-1} + \underline{R_5s_2^{-2}s_1^2s_2^{-2}s_1^2} + \\
&\quad + R_5(s_1^{-1}s_2^{-2}s_1)s_1s_2^{-1}(s_2^{-1}s_1^2s_2) + \\
&\quad + R_5s_1(s_1^{-1}s_2^{-2}s_1)s_1s_2^{-1}(s_2^{-1}s_1^2s_2)s_2 + R_5^\times s_2^{-2}s_1^2s_2^{-2}s_1^2s_2^3 \\
&\in u_1s_2^{-1}s_1s_2^2(s_1^{-1}s_2^3s_1)s_1s_2^{-1} + u_1s_2s_1^{-1}(s_1^{-1}s_2^{-1}s_1)(s_2^{-1}s_1s_2)s_2s_1^{-1} + \\
&\quad + u_1s_2s_1^{-1}(s_1^{-1}s_2^{-1}s_1)(s_2^{-1}s_1s_2)s_2s_1^{-1}s_2 + u_1^\times s_2^{-2}s_1^2s_2^{-2}s_1^2s_2^3 + U'''' \\
&\in u_1\Phi(s_2s_1^{-1}u_2u_1s_2s_1^{-1}s_2) + \underline{\underline{u_1s_2s_1^{-1}s_2(s_1^{-1}s_2^{-1}s_1)s_2s_1^{-1}s_2s_1^{-1}}} + \\
&\quad + u_1s_2s_1^{-1}s_2(s_1^{-1}s_2^{-1}s_1)s_2s_1^{-1}s_2s_1^{-1}s_2 + u_1^\times s_2^{-2}s_1^2s_2^{-2}s_1^2s_2^3 + U'''' \\
&\overset{4.4}{\in} u_1\Phi\big(s_2s_1^{-1}\omega^2u_1 + s_2s_1^{-1}s_2^2s_1^{-2}s_2s_1^{-1}s_2 + s_2s_1^{-1}u_1u_2u_1u_2u_1\big) + \\
&\quad + u_1s_2s_1^{-1}s_2^2s_1^{-2}s_2s_1^{-1}s_2 + u_1^\times s_2^{-2}s_1^2s_2^{-2}s_1^2s_2^3 + U''''
\end{aligned}$$

However, by Lemma 4.3(ii) and Proposition 4.8(i) we have that $\Phi\big(s_2s_1^{-1}\omega^2u_1 + s_2s_1^{-1}s_2^2s_1^{-2}s_2s_1^{-1}s_2 + \underline{\underline{s_2s_1^{-1}u_1u_2u_1u_2u_1}}\big) \subset \Phi(U'''') \overset{4.1}{\subset} U''''$. Therefore, it will be sufficient to prove that the element $s_2^{-2}s_1^2s_2^{-2}s_1^2s_2^3$ is inside $u_1^\times\omega^5 + U''''$. We expand $\mathbf{s_2^{-2}}$ as a linear combination of s_2^{-1}, 1, s_2, s_2^2 and s_2^3, where the coefficient of s_2^3 is invertible, and we have: $s_2^{-2}s_1^2\mathbf{s_2^{-2}}s_1^2s_2^3 \in u_1s_2^{-3}(s_2s_1^2s_2^{-1})s_1^2s_2^3 + \underline{u_1s_2^{-2}s_1^4s_2^3} + \underline{\underline{u_1s_2^{-2}(s_1^2s_2s_1)s_1s_2^3}} + u_1s_2^{-1}(s_2^{-1}s_1^2s_2)s_2s_1^2s_2^3 + u_1^\times s_2^{-2}s_1^2s_2^3s_1^2s_2^3$. By Lemma 4.9(i) we have that $u_1^\times s_2^{-2}s_1^2s_2^3s_1^2s_2^3 \subset u_1^\times\omega^5 + U''''$. Therefore, $s_2^{-2}s_1^2s_2^{-2}s_1^2s_2^3 \in \underline{\underline{u_1(s_1s_2^{-3}s_1^{-1})s_2^2s_1^3s_2^3}} + u_1s_2^{-1}s_1s_2^2s_1^{-1}s_2s_1^2s_2^3 + u_1^\times\omega^5 + U''''$. It remains to prove that the element $s_2^{-1}s_1s_2^2s_1^{-1}s_2s_1^2s_2^3$ is inside U''''. Indeed, $s_2^{-1}s_1s_2^2s_1^{-1}s_2s_1^2s_2^3 =$

$(s_2^{-1}s_1s_2)s_2(s_1^{-1}s_2s_1)s_1s_2^3 = s_1s_2(s_1^{-1}s_2^2s_1)(s_2^{-1}s_1s_2)s_2^2 = s_1s_2^2s_1^2(s_2^{-1}s_1s_2)s_1^{-1}s_2^2 = s_1s_2^3(s_2^{-1}s_1^3s_2)s_1^{-2}s_2^2 = \underline{\underline{s_1^2(s_1^{-1}s_2^3s_1)s_2^3s_1^{-3}s_2^2}}$.

(ii) As we explained in the beginning of this section, since $1 \in U$ it will be sufficient to prove that U is invariant under left multiplication by s_2. We use the fact that U is equal to the RHS of (i) and by the definition of U'''' we have:

$$U = U' + \sum_{k=2}^{4}\omega^k u_1 + \sum_{k=2}^{5}\omega^{-k}u_1 + u_1s_2^{-2}s_1^2s_2^{-1}s_1s_2^{-1}u_1 + u_1s_2^2s_1^{-2}s_2s_1^{-1}s_2u_1 + \\ + u_1s_2s_1^{-2}s_2^2s_1^{-2}s_2^2u_1 + u_1s_2^{-1}s_1^2s_2^{-2}s_1^2s_2^{-2}u_1.$$

On one hand, $s_2(U' + \omega^2u_1 + u_1s_2^{-2}s_1^2s_2^{-1}s_1s_2^{-1}u_1 + u_1s_2^2s_1^{-2}s_2s_1^{-1}s_2u_1) \subset U$ (Proposition 4.6, Lemma 4.3(ii) and Proposition 4.8(i), (ii)). On the other hand,

$$\sum_{k=2}^{5}s_2\omega^{-k}u_1 = \sum_{k=2}^{5}s_1^{-2}s_2^{-1}\omega^{-k+1}u_1 \\ \subset \Phi(\sum_{k=2}^{5}u_1s_2\omega^{k-1}u_1) \overset{4.2 \text{ and } 4.3\text{(ii)}}{\subset} u_1\Phi(U + s_2\omega^3 + s_2\omega^4)u_1$$

Therefore, by Lemma 4.1 we only need to prove that

$$s_2(\omega^3u_1 + \omega^4u_1 + u_1s_2s_1^{-2}s_2^2s_1^{-2}s_2^2u_1 + u_1s_2^{-1}s_1^2s_2^{-2}s_1^2s_2^{-2}u_1) \subset U.$$

We first notice that $s_2\omega^4u_1 = s_2\omega^3\omega u_1 = s_2\omega^3u_1\omega$. Therefore, in order to prove that $s_2(\omega^3u_1 + \omega^4u_1) \subset U$, it will be sufficient to prove that $s_2\omega^3u_1 \subset u_1s_2u_1u_2u_1u_2u_1$ (Propositions 4.6 and 4.8(iii)). Indeed, we have: $s_2\omega^3u_1 = s_2\omega^2\omega u_1 \overset{4.3\text{(ii)}}{=} s_1s_2s_1^4s_2s_1^3s_2\omega u_1 = s_1s_2s_1^4s_2s_1^4(s_1^{-1}s_2^2s_1)s_1s_2u_1 = s_1s_2s_1^4s_2s_1^4s_2s_1^2(s_2^{-1}s_1s_2)u_1 \subset u_1s_2u_1(s_2u_1s_2u_1s_2u_1)$. However, by Lemma 4.3(i) we have that $u_1s_2u_1(s_2u_1s_2u_1s_2u_1) \subset u_1s_2u_1(\omega^2u_1 + u_1u_2u_1u_2u_1)$. The result follows from Lemma 4.3(ii).

It remains to prove that $s_2u_1s_2^{-1}s_1^2s_2^{-2}s_1^2s_2^{-2}u_1$ and $s_2u_1s_2s_1^{-2}s_2^2s_1^{-2}s_2^2u_1$ are subsets of U. We have:

$$s_2u_1s_2^{-1}s_1^2s_2^{-2}s_1^2s_2^{-2} = s_2(R_5 + R_5s_1 + R_5s_1^{-1} + R_5s_1^2 + R_5s_1^{-2})s_2^{-1}s_1^2s_2^{-2}s_1^2s_2^{-2} \\ \subset \underline{R_5s_1^2s_2^{-2}s_1^2s_2^{-2}} + \underline{\underline{R_5(s_2s_1s_2^{-1})s_1^2s_2^{-2}s_1^2s_2^{-2}}} + \\ + \underline{\underline{R_5(s_2s_1^{-1}s_2^{-1})s_1^2s_2^{-2}s_1^2s_2^{-2}}} + R_5(s_2s_1^2s_2^{-1})s_1^2s_2^{-2}s_1^2s_2^{-2} + \\ + R_5(s_2s_1^{-2}s_2^{-1})s_1^2s_2^{-2}s_1^2s_2^{-2} \\ \subset u_1s_2^2s_1^3s_2^{-2}s_1^2s_2^{-2} + u_1s_2^{-2}u_1s_2^{-2}s_1^2s_2^{-2} + U.$$

However, by Lemma 4.9(iii) we have that $u_1 s_2^{-2} u_1 s_2^{-2} s_1^2 s_2^{-2} \subset U$. Therefore, it remains to prove that the element $s_2^2 s_1^3 s_2^{-2} s_1^2 s_2^{-2}$ is inside U. For this purpose, we expand s_1^3 as a linear combination of s_1^2, s_1, 1, s_1^{-1} and s_1^{-2} and we have:

$$\begin{aligned} s_2^2 s_1^3 s_2^{-2} s_1^2 s_2^{-2} \in\ & R_5 s_2^2 s_1^2 s_2^{-2} s_1^2 s_2^{-2} + \underline{\underline{R_5 s_2^2 (s_1 s_2^{-2} s_1^{-1}) s_1^3 s_2^{-2}}} + \underline{u_1 s_2^{-2}} + \\ & + \underline{\underline{R_5 s_1^{-1} (s_1 s_2^2 s_1^{-1}) s_2^{-2} s_1^2 s_2^{-2}}} + R_5 s_2^2 s_1^{-2} s_2^{-2} s_1^2 s_2^{-2}. \end{aligned}$$

However, $s_2^2 s_1^2 s_2^{-2} s_1^2 s_2^{-2} = s_2 (s_2 s_1^2 s_2^{-1}) s_2^{-1} s_1 (s_1 s_2^{-2} s_1^{-1}) s_1 = s_2 s_1^{-1} s_2 (s_2 s_1 s_2^{-1}) s_1 s_2^{-1} s_1^{-2} s_2 s_1 = s_2 s_1^{-2} (s_1 s_2 s_1^{-1})(s_2 s_1^2 s_2^{-1}) s_1^{-2} s_2 s_1 = s_2 s_1^{-2} s_2^{-1} (s_1 s_2 s_1^{-1}) s_2^2 s_1^{-1} s_2 s_1 = \underline{\underline{u_1 s_2 s_1^{-2} s_2^{-2} (s_1 s_2^3 s_1^{-1}) s_2 s_1}}$. Moreover, we expand s_1^2 as a linear combination of s_1, 1, s_1^{-1}, s_1^{-2} and s_1^{-3} and we have:

$$\begin{aligned} s_2^2 s_1^{-2} s_2^{-2} s_1^2 s_2^{-2} \ \in\ & \underline{R_5 s_2^2 s_1^{-2} s_2^{-4}} + \underline{\underline{R_5 s_1^{-1} (s_1 s_2^2 s_1^{-1})(s_1^{-1} s_2^{-2} s_1) s_2^{-2}}} + \\ & + R_5 s_2^2 s_1^{-2} s_2^{-1} (s_2^{-1} s_1^{-1} s_2^{-2}) + R_5 s_2 (s_2 s_1^{-2} s_2^{-1}) \omega^{-1} s_2^{-1} + \\ & + \Phi(s_2^{-2} s_1^2 s_2^2 s_1^3 s_2^2) \\ \overset{4.7}{\in}\ & R_5 s_2^2 s_1^{-2} \omega^{-1} s_1^{-1} + R_5 s_2 s_1^{-1} s_2^{-2} s_1 \omega^{-1} s_2^{-1} + \Phi(U) + U \\ \overset{4.1}{\subset}\ & \underline{s_2^2 \omega^{-1} u_1} + R_5 s_2 s_1^{-1} s_2^{-2} \omega^{-1} s_1 s_2^{-1} + U \\ \subset\ & s_2 s_1^{-1} u_2 u_1 s_2^{-1} s_1 s_2^{-1} + U. \end{aligned}$$

Therefore, by Proposition 4.8(ii) we have that the element $s_2^2 s_1^{-2} s_2^{-2} s_1^2 s_2^{-2}$ is inside U and, hence, $s_2 u_1 s_2^{-1} s_1^2 s_2^{-2} s_1^2 s_2^{-2} u_1 \subset U$.

In order to finish the proof that $H_5 = U$ it remains to prove that $s_2 u_1 s_2 s_1^{-2} s_2^2 s_1^{-2} s_2^2 \subset U$. For this purpose we expand u_1 as $R_5 + R_5 s_1 + R_5 s_1^2 + R_5 s_1^3 + R_5 s_1^4$ and we have:

$$\begin{aligned} s_2 u_1 s_2 s_1^{-2} s_2^2 s_1^{-2} s_2^2 \subset\ & \Phi(s_2^{-2} u_1 s_2^{-2} s_1^2 s_2^{-2}) + \underline{\underline{R_5 (s_2 s_1 s_2) s_1^{-2} s_2^2 s_1^{-2} s_2^2}} + \\ & + R_5 \omega s_1^{-2} s_2^2 s_1^{-2} s_2^2 + + R_5 s_2 s_1^3 s_2 s_1^{-2} s_2^2 s_1^{-2} s_2^2 + \\ & + R_5 s_2 s_1^4 s_2 s_1^{-2} s_2^2 s_1^{-2} s_2^2. \end{aligned}$$

However, by Lemma 4.9(iii) we have that $\Phi(s_2^{-2} u_1 s_2^{-2} s_1^2 s_2^{-2}) \subset \Phi(U) \overset{4.1}{\subset} U$. Moreover, $\omega s_1^{-2} s_2^2 s_1^{-2} s_2^2 = \underline{\underline{R_5 s_1^{-2} \omega s_2^2 s_1^{-2} s_2^2}}$. It remains to prove that $C := R_5 s_2 s_1^3 s_2 s_1^{-2} s_2^2 s_1^{-2} s_2^2 + R_5 s_2 s_1^4 s_2 s_1^{-2} s_2^2 s_1^{-2} s_2^2$ is a subset of U. We have:

$$\begin{aligned} C &= R_5 s_2 s_1^2 (s_1 s_2 s_1^{-1}) s_1^{-1} s_2 (s_2 s_1^{-2} s_2^{-1}) s_2^3 + R_5 s_1^{-1} (s_1 s_2 s_1^4) s_2 s_1^{-2} s_2 (s_2 s_1^{-2} s_2^{-1}) s_2^3 \\ &= R_5 (s_2 s_1^2 s_2^{-1})(s_1 s_2 s_1^{-1}) s_2 s_1^{-1} s_2^{-2} s_1 s_2^3 + R_5 s_1^{-1} s_2^4 (s_1 s_2 s_1^{-1}) s_1^{-1} (s_2 s_1^{-1} s_2^{-1}) s_2^{-1} s_1 s_2^3 \\ &= R_5 s_1^{-1} s_2 (s_2 s_1 s_2^{-1})(s_1 s_2^2 s_1^{-1}) s_2^{-2} s_1 s_2^3 + R_5 s_1^{-1} s_2^2 \omega s_1^{-2} s_2^{-1} s_1 s_2^{-1} s_1 s_2^3 \\ &= R_5 s_1^{-1} s_2 s_1^{-1} (s_2 s_1 s_2^{-1}) s_1^2 s_2^{-1} s_1 s_2^3 + R_5 s_1^{-1} s_2^2 s_1^{-2} (s_2 s_1^3 s_2^{-1}) s_1 s_2^3 + U \\ &= \underline{\underline{R_5 s_1^{-1} s_2 s_1^{-2} (s_2 s_1^3 s_2^{-1}) s_1 s_2^3}} + u_1 s_2^2 s_1^{-3} s_2^3 s_1^2 s_2^3 + U. \end{aligned}$$

We expand s_1^{-3} as a linear combination of s_1^{-2}, s_1^{-1}, 1, s_1, s_1^2 and s_1^3 and we have that

$$u_1s_2^2s_1^{-3}s_2^3s_1^2s_2^3 \subset u_1s_2^2s_1^{-2}s_2^3s_1^2s_2^3 + \underline{\underline{u_1(s_1s_2^2s_1^{-1})s_2^3s_1^2s_2^3}} + \underline{u_1s_2^5s_1^2s_2^3} + \\ + \underline{\underline{u_1s_2(s_2s_1s_2^3)s_1^2s_2^3}} + u_1s_2^2s_1^3s_2^3s_1^2s_2^3.$$

Hence, in order to finish the proof that $H_5 = U$ we have to prove that $u_1s_2^2s_1^{-2}s_2^3s_1^2s_2^3$ and $u_1s_2^2s_1^2s_2^3s_1^2s_2^3$ are subsets of U. We have:

$$\begin{aligned} u_1s_2^2s_1^{-2}s_2^3s_1^2s_2^3 &= u_1s_2^2s_1^{-2}(as_2^2+bs_2+c+ds_2^{-1}+es_2^{-2})s_1^2s_2^3 \\ &\subset u_1s_2^2s_1^{-2}s_2^2s_1^2s_2^3 + u_1s_2^2s_1^{-2}s_2^2s_1^2s_2^3 + \underline{u_1s_2^5} + \underline{\underline{u_1s_2(s_2s_1^{-2}s_2^{-1})s_1^2s_2^3}} + \\ &\quad + u_1s_2^2s_1^{-2}s_2^{-2}s_1^2s_2^3 \end{aligned}$$

However, we have that $u_1s_2^2s_1^{-2}s_2^2s_1^2s_2^3 = \underline{\underline{u_1(s_1^{-1}s_2^2s_1)s_2(s_2^{-1}s_1^{-3}s_2)s_1^2s_2^3}}$. Moreover, we have $u_1s_2^2s_1^{-2}s_2^{-2}s_1^2s_2^3 = u_1(s_1s_2^2s_1^{-1})(s_1^{-1}s_2^{-2}s_1)s_1s_2^3 = u_1s_2^{-1}s_1^2s_2^3\omega^{-1}s_1s_2^3 = u_1s_2^{-1}s_1^2s_2^3s_1(s_2^{-1}s_1^{-2}s_2)s_2 = u_1s_2^{-1}s_1^2s_2^3s_1^2s_2^{-1}(s_2^{-1}s_1^{-1}s_2) \subset \Phi(s_2u_2u_1s_2s_1^{-1}s_2)$. By Proposition 4.8(i) and Lemma 4.1 we have that $u_1s_2^2s_1^{-2}s_2^{-2}s_1^2s_2^3 \subset U$. It remains to prove that $u_1s_2^2s_1^{-2}s_2^2s_1^2s_2^3 \subset U$. We notice that $u_1s_2^2s_1^{-2}s_2^2s_1^2s_2^3 = u_1s_2^3(s_2^{-1}s_1^{-2}s_2)s_2s_1^2s_2^3 = u_1(s_1^{-1}s_2^3s_1)s_2^{-2}s_1^{-1}s_2s_1^2s_2^3 = u_1s_2s_1^3s_2^{-3}s_1^{-1}s_2s_1^2s_2^3$. We expand s_2^3 as a linear combination of s_2^2, s_2, 1, s_2^{-1} and s_2^{-2} and we have:

$$\begin{aligned} u_1s_2s_1^3s_2^{-3}s_1^{-1}s_2s_1^2s_2^3 &\subset u_1s_2s_1^3s_2^{-2}(s_2^{-1}s_1^{-1}s_2)s_1^2s_2^2 + u_1s_2s_1^3s_2^{-3}s_1^{-1}\omega + \\ &\quad + \underline{\underline{u_1s_2s_1^3s_2^{-3}s_1^{-1}s_2s_1^2}} + \underline{\underline{u_1s_2s_1^3s_2^{-3}s_1^{-1}(s_2s_1^2s_2^{-1})}} + \\ &\quad + u_1s_2s_1^3s_2^{-3}s_1^{-1}(s_2s_1^2s_2^{-1})s_2^{-1} \\ &\subset u_1s_2s_1^3s_2^{-2}s_1(s_2^{-1}s_1s_2)s_2 + u_1s_2s_1^3s_2^{-3}s_1^{-2}s_2(s_2s_1s_2^{-1}) + \\ &\quad + \underline{\underline{u_1s_2s_1^3s_2^{-3}\omega s_1^{-1}}} + U \\ &\subset u_1(s_2u_1u_2u_1s_2s_1^{-1}s_2)u_1 + U. \end{aligned}$$

The result follows from Proposition 4.8(i) and Lemma 4.1.
Using analogous calculations we will prove that $u_1s_2^2s_1^2s_2^3s_1^2s_2^3$ is a subset of U. We have:

$$\begin{aligned} u_1s_2^2s_1^2s_2^3s_1^2s_2^3 &= u_1s_2^2s_1^2(as_2^2+bs_2+c+ds_2^{-1}+es_2^{-2})s_1^2s_2^3 \\ &\subset u_1s_2^2s_1^2s_2^2s_1^2s_2^3 + u_1s_2\omega s_1^2s_2^3 + \underline{u_1s_2^2s_1^4s_2^3} + \underline{\underline{u_1s_2(s_2s_1^2s_2^{-1})s_1^2s_2^3}} + \\ &\quad + u_1s_2^2s_1^2s_2^{-2}s_1^2s_2^3. \end{aligned}$$

However, $u_1s_2\omega s_1^2s_2^3 = \underline{\underline{u_1s_2s_1^2\omega s_2^3}}$. Therefore, it remains to prove that $u_1s_2^2s_1^2s_2^2s_1^2s_2^3$ and $u_1s_2^2s_1^2s_2^{-2}s_1^2s_2^3$ are subsets of U. We have:

$$u_1s_2^2s_1^2s_2^2s_1^2s_2^3 = u_1s_2^2s_1^2s_2^2s_1^2(as_2^2+bs_2+c+ds_2^{-1}+es_2^{-2}) \subset u_1s_2\omega^2s_2+u_1s_2\omega^2+\underline{u_1s_2^2s_1^2s_2^2s_1^2}+u_1s_2\omega s_2s_1^2s_2^{-1}+u_1s_2\omega s_2s_1^2s_2^{-2}$$

By Lemma 4.3(ii) we have that $u_1s_2\omega^2s_2+u_1s_2\omega^2 \subset u_1s_2s_1^4s_2s_1^3s_2u_1s_2+U$. However, by Lemma 4.3(i) we have $u_1s_2s_1^4(s_2s_1^3s_2u_1s_2) \subset u_1s_2s_1^4(\omega^2u_1+u_1u_2u_1u_2u_1) \subset u_1s_2\omega^2u_1 + \underline{\underline{u_1s_2u_1u_2u_1u_2u_1}}$. Using another time Lemma 4.3(ii) we have that $u_1s_2s_1^4(\omega^2u_1+u_1u_2u_1u_2u_1) \subset U$. It remains to prove that $D := u_1s_2\omega s_2s_1^2s_2^{-1} + u_1s_2\omega s_2s_1^2s_2^{-2}$ is a subset of U. We have:

$$\begin{aligned} D &= u_1s_2\omega(s_2s_1^2s_2^{-1})+u_1s_2\omega(s_2s_1^2s_2^{-1})s_2^{-1} \\ &= u_1s_2\omega s_1^{-1}s_2^2s_1+u_1s_2\omega s_1^{-1}s_2^2s_1s_2^{-1} \\ &= \underline{\underline{u_1s_2s_1^{-1}\omega s_2^2s_1}}+u_1s_2s_1^{-1}\omega s_2^2s_1s_2^{-1}. \end{aligned}$$

However, we have $u_1s_2s_1^{-1}\omega s_2^2s_1s_2^{-1} = u_1s_2s_1^{-1}\omega s_2(s_2s_1s_2^{-1}) = u_1s_2s_1^{-1}s_2s_1(s_1s_2s_1^{-1})s_2s_1 = \underline{\underline{u_1s_2s_1^{-1}(s_2s_1s_2^{-1})s_1s_2^2s_1}}$, meaning that $D \subset U$. Using analogous calculations we will prove that $u_1s_2^2s_1^2s_2^{-2}s_1^2s_2^3 \subset U$. We have:

$$\begin{aligned} u_1s_2^2s_1^2s_2^{-2}s_1^2s_2^3 &= u_1s_2(s_2s_1^2s_2^{-1})s_2^{-1}s_1^2s_2^3 \\ &= u_1s_2s_1^{-1}s_2^2s_1s_2^{-1}s_1^2(as_2^2+bs_2+c+ds_2^{-1}+es_2^{-2}) \\ &\subset u_1(s_1s_2s_1^{-1})s_2(s_2s_1s_2^{-1})s_1^2s_2^2+\underline{\underline{u_1s_2s_1^{-1}s_2^2s_1(s_2^{-1}s_1^2s_2)}}+ \\ &\quad +\underline{\underline{u_1s_2s_1^{-1}s_2^2s_1s_2^{-1}s_1^2}}+u_1s_2s_1^{-1}s_2(s_2s_1s_2^{-1})s_1^2s_2^{-1}+ \\ &\quad +u_1s_2s_1^{-1}s_2(s_2s_1s_2^{-1})s_1^2s_2^{-2} \\ &\subset u_1s_2^{-1}(s_1s_2^2s_1^{-1})s_2s_1^3s_2^2+\underline{\underline{u_1s_2s_1^{-1}s_2s_1^{-1}(s_2s_1^3s_2^{-1})}}+ \\ &\quad +u_1s_2s_1^{-2}(s_1s_2s_1^{-1})(s_2s_1^3s_2^{-1})s_2^{-1}+U \\ &\subset u_1s_2^{-2}s_1^2s_2^2s_1^3s_2^2+u_1s_2s_1^{-2}s_2^{-1}(s_1s_2s_1^{-1})s_2^2(s_2s_1s_2^{-1})+U \\ &\overset{4.7}{\subset} \underline{\underline{u_1s_2s_1^{-2}s_2^{-2}(s_1s_2^3s_1^{-1})s_2s_1}}+U. \qquad \square \end{aligned}$$

Corollary 4.11. *H_5 is a free R_5-module of rank $r_5 = 600$.*

Proof. By Proposition 2.2 it will be sufficient to show that H_5 is generated as R_5-module by r_5 elements. By Theorem 4.10 the definition of U'''' and the fact that $u_1u_2u_1 = u_1+u_1s_2u_1+u_1s_2^{-1}u_1+u_1s_2^2u_1+u_1s_2^{-2}u_1$ we have that H_5 is spanned as left u_1-module by 120 elements. Since u_1 is spanned by 5 elements as a R_5-module, we have that H_5 is spanned over R by $r_5 = 600$ elements. $\square$

5. The irreducible representations of B_3 of dimension at most 5

We set $\tilde{R}_k = \mathbb{Z}[u_1^{\pm 1}, ..., u_k^{\pm 1}]$, $k = 2, 3, 4, 5$. Let $\tilde{H}_k$ denote the quotient of the group algebra $\tilde{R}_k B_3$ by the relations $(s_i - u_1)...(s_i - u_k)$, $i = 1, 2$. In the previous sections we proved that H_k is a free R_k-module of rank r_k. Hence, $\tilde{H}_k$ is a free $\tilde{R}_k$-module of rank r_k (Lemma 2.3 in [16]). We now assume that $\tilde{H}_k$ has a unique symmetrizing trace $t_k : \tilde{H}_k \to \tilde{R}_k$ (i.e. a trace function such that the bilinear form $(h, h') \mapsto t_k(hh')$ is non-degenerate), having nice properties (see [3], Theorem 2.1): for example, $t_k(1) = 1$, which means that t_k specializes to the canonical symmetrizing form on $\mathbb{C}W_k$.

Let μ_∞ be the group of all roots of unity in $\mathbb{C}$. We recall that W_k is the finite quotient group $B_3/\langle s_i^k \rangle$, $k = 2, 3, 4, 5$ and $i = 1, 2$. We denote by K_k the *field of definition* of W_k, i.e. the number field contained in $\mathbb{Q}(\mu_\infty)$, which is generated by the traces of all elements of W_k (for more details see [1]). We denote by $\mu(K_k)$ the group of all roots of unity of K_k and, for every integer $m > 1$, we set $\zeta_m := \exp(2\pi i/m)$, where i denotes here a square root of -1.

Let $\mathbf{v} = (v_1, ..., v_k)$ be a set of k indeterminates such that, for every $i \in \{1, ..., k\}$, we have $v_i^{|\mu(K_k)|} = \zeta_k^{-i} u_i$. By extension of scalars we obtain a $\mathbb{C}(\mathbf{v})$-algebra $\mathbb{C}(\mathbf{v})\tilde{H}_k := \tilde{H}_k \otimes_{\tilde{R}_k} \mathbb{C}(\mathbf{v})$, which is split semisimple (see [12], Theorem 5.2). Since the algebra $\mathbb{C}(\mathbf{v})\tilde{H}_k$ is split, by Tits' deformation theorem (see Theorem 7.4.6 in [9]), the specialization $v_i \mapsto 1$ induces a bijection $\mathrm{Irr}(\mathbb{C}(\mathbf{v})\tilde{H}_k) \to \mathrm{Irr}(W_k)$.

Let $\varrho : B_3 \to GL_n(\mathbb{C})$ be an irreducible representation of B_3 of dimension $k \leq 5$. We set $A := \varrho(s_1)$ and $B := \varrho(s_2)$. The matrices A and B are similar since s_1 and s_2 are conjugate ($s_2 = (s_1 s_2) s_1 (s_1 s_2)^{-1}$). Hence, by Cayley–Hamilton theorem of linear algebra, there is a monic polynomial $m(X) = X^k + m_{n-1}X^{n-1} + ... + m_1 X + m_0 \in \mathbb{C}[X]$ of degree k such that $m(A) = m(B) = 0$. Let R_K^k denotes the integral closure of R_k in K_k. We fix $\theta : R_K^k \to \mathbb{C}$ a *specialization* of R_K^k, defined by $u_i \mapsto \lambda_i$, where λ_i are the eigenvalues of A (and B). We notice that θ is well-defined, since $m_0 = \det A \in \mathbb{C}^\times$. Therefore, in order to determine ϱ it will be sufficient to describe the irreducible $\mathbb{C}\tilde{H}_k := \tilde{H}_k \otimes_\theta \mathbb{C}$-modules of dimension k.

When the algebra $\mathbb{C}\tilde{H}_k$ is semisimple, we can use again Tits' deformation theorem and we have a canonical bijection between the set of irreducible characters of $\mathbb{C}\tilde{H}_k$ and the set of irreducible characters of $\mathbb{C}(\mathbf{v})\tilde{H}_k$, which are in bijection with the irreducible characters of W_k. However, this is not always the case. In order to determine the irreducible representations of $\mathbb{C}\tilde{H}_k$ in the general case (when we don't know a priori that $\mathbb{C}\tilde{H}_k$ is semisimple) we use a different approach.

Let $R_0^+(\mathbb{C}(\mathbf{v})\tilde{H}_k)$ (respectively $R_0^+(\mathbb{C}\tilde{H}_k)$) denote the subset of the *Grothendieck group* of the category of finite dimensional $\mathbb{C}(\mathbf{v})\tilde{H}_k$ (respectively $\mathbb{C}\tilde{H}_k$)-modules consisting of elements $[V]$, where V is a $\mathbb{C}(\mathbf{v})\tilde{H}_k$ (respectively $\mathbb{C}\tilde{H}_k$)-module (for more details, one may refer to §7.3 in [9]). By Theorem 7.4.3 in [9] we obtain a well-defined decomposition map

$$d_\theta : R_0^+(\mathbb{C}(\mathbf{v})\tilde{H}_k) \to R_0^+(\mathbb{C}\tilde{H}_k).$$

The corresponding *decomposition matrix* is the $\mathrm{Irr}\left(\mathbb{C}(\mathbf{v})\tilde{H}_k\right) \times \mathrm{Irr}(\mathbb{C}\tilde{H}_k)$ matrix $(d_{\chi\phi})$ with non-negative integer entries such that $d_\theta([V_\chi]) = \sum_\phi d_{\chi\phi}[V'_\phi]$, where V_χ is an irreducible $\mathbb{C}(\mathbf{v})\tilde{H}_k$-module with character χ and V_ϕ is an irreducible $\mathbb{C}\tilde{H}_k$-module with character ϕ. This matrix records in which way the irreducible representations of the semisimple algebra $\mathbb{C}(\mathbf{v})\tilde{H}_k$ break up into irreducible representations of $\mathbb{C}\tilde{H}_k$.

The form of the decomposition matrix is controlled by the *Schur elements*, denoted as s_χ, $\chi \in \mathrm{Irr}\left(\mathbb{C}(\mathbf{v})\tilde{H}_k\right)$, with respect to the symmetric form t_k. The Schur elements belong to R_K^k (see [9], Proposition 7.3.9) and they depend only on the symmetrizing form t_k and the isomorphism class of the representation. Moreover, M. Chlouveraki has shown that these elements are products of cyclotomic polynomials over K_k evaluated on monomials of degree 0 (see Theorem 4.2.5 in [5]). In the following section we are going to use these elements in order to determine the irreducible representations of $\mathbb{C}\tilde{H}_k$ (for more details about the definition and the properties of the Schur elements, one may refer to §7.2 in [9]).

We say that the $\mathbb{C}(\mathbf{v})\tilde{H}_k$-modules V_χ, V_ψ *belong to the same block* if the corresponding characters χ, ψ label the rows of the same block in the decomposition matrix $(d_{\chi\phi})$ (by definition, this means that there is a $\phi \in \mathrm{Irr}(\mathbb{C}\tilde{H}_k)$ such that $d_{\chi,\phi} \neq 0 \neq d_{\psi,\phi}$). If an irreducible $\mathbb{C}(\mathbf{v})\tilde{H}_k$-module is alone in its block, then we call it a *module of defect 0*. Motivated by the idea of M. Chlouveraki and H. Miyachy in [6] §3.1 we use the following criteria in order to determine whether two modules belong to the same block:

- We have $\theta(s_\chi) \neq 0$ if and only if V_χ is a module of defect 0 (see [14], Lemma 2.6). This criterium together with Theorem 7.5.11 in [9], states that V_χ is a module of defect 0 if and only if the decomposition matrix is of the form

$$\begin{pmatrix} * & \dots & * & 0 & * & \dots & * \\ \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ * & \dots & * & 0 & * & \dots & * \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ * & \dots & * & 0 & * & \dots & * \\ \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ * & \dots & * & 0 & * & \dots & * \end{pmatrix}$$

- If V_χ, V_ψ are in the same block, then $\theta(\omega_\chi(z_0)) = \theta(\omega_\psi(z_0))$ (see [9], Lemma 7.5.10), where ω_χ, ω_ψ are the corresponding *central characters*[3] and z_0 is the central element $(s_1 s_2)^3$.

[3] If z lies in the center of $\mathbb{C}(\mathbf{v})\tilde{H}_k$ then Schur's lemma implies that z acts as scalars in V_χ and V_ψ. We denote these scalars as $\omega_\chi(z)$ and $\omega_\psi(z)$ and we call the associated $\mathbb{C}(\mathbf{v})$-homomorphisms ω_χ, ω_ψ : $Z(\mathbb{C}(\mathbf{v})\tilde{H}_k) \to \mathbb{C}(\mathbf{v})$ central characters (for more details, see [9] page 227).

We recall that in order to describe the irreducible representations of B_3 of dimension at most 5, it is enough to describe the irreducible $\mathbb{C}\tilde{H}_k$-modules of dimension k. Let S be an irreducible $\mathbb{C}\tilde{H}_k$-module of dimension k and $s \in S$ with $s \neq 0$. The morphism $f_s : \mathbb{C}\tilde{H}_k \to S$ defined by $h \mapsto hs$ is surjective since S is irreducible. Hence, by the definition of the Grothendieck group we have that $d_\theta\big([\mathbb{C}(\mathbf{v})\tilde{H}_k]\big) = [\mathbb{C}\tilde{H}_k] = [\ker f_s] + [S]$. However, since $\mathbb{C}(\mathbf{v})\tilde{H}_k$ is semisimple we have $\mathbb{C}(\mathbf{v})H_k = M_1 \oplus ... \oplus M_r$, where the M_i are (up to isomorphism) all the simple $\mathbb{C}(\mathbf{v})\tilde{H}_k$-modules (with redundancies). Therefore, we have $\sum\limits_{i=1}^{r} d_\theta([M_i]) = [\ker f_s] + [S]$. Hence, there is a simple $\mathbb{C}(\mathbf{v})\tilde{H}_k$-module M such that

$$d_\theta([M]) = [S] + [J], \tag{4}$$

where J is a $\mathbb{C}\tilde{H}_k$-module.

Remark 5.1.

(i) The $\mathbb{C}(\mathbf{v})\tilde{H}_k$-module M is of dimension at least k.

(ii) If J is of dimension 1, there is a $\mathbb{C}(\mathbf{v})\tilde{H}_k$-module N of dimension 1, such that $d_\theta([N]) = [J]$. This result comes from the fact that the 1-dimensional $\mathbb{C}\tilde{H}_k$-modules are of the form (λ_i), $i = 1, \dots, k$ and, by definition, $\lambda_i = \theta(u_i)$.

The irreducible $\mathbb{C}(\mathbf{v})\tilde{H}_k$-modules are known (see [13] or [2] §5B and §5D, for $n = 3$ and $n = 4$, respectively). Therefore, we can determine S by using (4) and a case-by-case analysis.

- $\underline{k=2}$: Since $\tilde{H}_2$ is the generic Hecke algebra of $\mathfrak{S}_3$, which is a Coxeter group, the irreducible representations of $\mathbb{C}\tilde{H}_2$ are well-known; we have two irreducible representations of dimension 1 and one of dimension 2. By (4) and Remark 5.1 (i), M must be the irreducible $\mathbb{C}(\mathbf{v})\tilde{H}_k$-module of dimension 2 and (4) becomes $[S] = d_\theta([M])$. Hence, we have:

$$A = \begin{bmatrix} \lambda_1 & \lambda_1 \\ 0 & \lambda_2 \end{bmatrix},\ B = \begin{bmatrix} \lambda_2 & 0 \\ -\lambda_2 & \lambda_1 \end{bmatrix}$$

  Moreover, $[S] = d_\theta([M])$ is irreducible and M is the only irreducible $\mathbb{C}(\mathbf{v})\tilde{H}_k$-module of dimension 2. As a result, M has to be alone in its block i.e. $\theta(s_\chi) \neq 0$, where χ is the character that corresponds to M. Therefore, an irreducible representation of B_3 of dimension 2 can be described by the explicit matrices A and B we have above, depending only on a choice of λ_1, λ_2 such that $\theta(s_\chi) = \lambda_1^2 - \lambda_1\lambda_2 + \lambda_2^2 \neq 0$.
- $\underline{k=3}$: Since the algebra $\mathbb{C}(\mathbf{v})\tilde{H}_3$ is split, we have a bijection between the set $\mathrm{Irr}(\mathbb{C}(\mathbf{v})\tilde{H}_3)$ and the set $\mathrm{Irr}(W)$, as we explained earlier. We refer to J. Michel's version of CHEVIE package of GAP3 (see [17]) in order to find the irreducible characters of W_3. We type:

```
gap> W_3:=ComplexReflectionGroup(4);
gap> CharNames(W_3);
[ "phi{1,0}", "phi{1,4}", "phi{1,8}", "phi{2,5}", "phi{2,3}",
"phi{2,1}", "phi{3,2}" ]
```

We have 7 irreducible characters $\phi_{i,j}$, where i is the dimension of the representation and j the valuation of its fake degree (see [12] §6A). Since S is of dimension 3, the equation (4) becomes $[S] = d_\theta([M])$, where M is the irreducible $\mathbb{C}(\mathbf{v})\tilde{H}_3$-module that corresponds to the character $\phi_{3,2}$ (see Remark 5.1(i)). However, we have explicit matrix models for this representation (see [2], §5B or we can refer to CHEVIE package of GAP3 again) and since $[S] = d_\theta([M])$ we have:

$$A = \begin{bmatrix} \lambda_3 & 0 & 0 \\ \lambda_1\lambda_3 + \lambda_2^2 & \lambda_2 & 0 \\ \lambda_2 & 1 & \lambda_1 \end{bmatrix},\ B = \begin{bmatrix} \lambda_1 & -1 & \lambda_2 \\ 0 & \lambda_2 & -\lambda_1\lambda_3 - \lambda_2^2 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

M is the only irreducible $\mathbb{C}(\mathbf{v})\tilde{H}_3$-module of dimension 3, therefore, as in the case where $k = 2$, we must have that $\theta(s_{\phi_{3,2}}) \neq 0$. The Schur element $s_{\phi_{3,2}}$ has been determined in [11] and the condition $\theta(s_{\phi_{3,2}}) \neq 0$ becomes

$$\frac{(\lambda_1^2 + \lambda_2\lambda_3)(\lambda_2^2 + \lambda_1\lambda_3)(\lambda_3^2 + \lambda_1\lambda_2)}{(\lambda_1\lambda_2\lambda_3)^2} \neq 0. \tag{5}$$

To sum up, an irreducible representation of B_3 of dimension 3 can be described by the explicit matrices A and B we gave above, depending only on a choice of λ_1, λ_2, λ_3 such that (5) is satisfied.

- $\underline{k = 4}$: We use again the program GAP3 package CHEVIE in order to find the irreducible characters of W_4. In this case we have 16 irreducible characters among which 2 of dimension 4; the characters $\phi_{4,5}$ and $\phi_{4,3}$ (we follow again the notations in GAP3, as in the case where $k = 3$).
  By Remark 5.1(i) and relation (4), we have $[S] = d_\theta([M])$, where M is the irreducible $\mathbb{C}(\mathbf{v})\tilde{H}_4$-module that corresponds either to the character $\phi_{4,5}$ or to the character $\phi_{4,3}$. We have again explicit matrix models for these representations (see [2], §5D, where we multiply the matrices described there by a scalar t and we set $u_1 = t, u_2 = tu, u_3 = tv$ and $u_4 = tw$):

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ \frac{\lambda_1^2}{\lambda_2} & \lambda_2 & 0 & 0 \\ \frac{\lambda_1^3}{r} & \frac{\lambda_1\lambda_2\lambda_3 - \lambda_1 r}{r} & \lambda_3 & 0 \\ -\lambda_2 & \lambda_2\alpha & \frac{r\alpha}{\lambda_1^2} & \lambda_4 \end{bmatrix},\ B = \begin{bmatrix} \lambda_4 & \lambda_3\alpha & \frac{\lambda_2\lambda_3\alpha}{\lambda_1} & -\frac{\lambda_2\lambda_3^2}{r} \\ 0 & \lambda_3 & \frac{\lambda_2\lambda_3 - r}{\lambda_1} & \frac{\lambda_1^2\lambda_3}{r} \\ 0 & 0 & \lambda_2 & \frac{\lambda_1^3}{r} \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix},$$

where $r := \pm\sqrt{\lambda_1\lambda_2\lambda_3\lambda_4}$ and $\alpha := \frac{r-\lambda_2\lambda_3-\lambda_1\lambda_4}{\lambda_1^2}$.
Since $d_\theta([M])$ is irreducible either M is of defect 0 or it is in the same block as the other irreducible module of dimension 4 i.e. $\theta(\omega_{\phi_{4,5}}(z_0)) = \theta(\omega_{\phi_{4,3}}(z_0))$. We use the program GAP3 package CHEVIE in order to calculate these central characters.
More precisely, we have 16 representations where the last 2 are of dimension 4. These representations will be noted in GAP3 as `R[15]` and `R[16]`. Since $z_0 = (s_1s_2)^3$ we need to calculate the matrices $R[i](s_1s_2s_1s_2s_1s_2), i = 15, 16$. These are the matrices `Product(R[15]{[1,2,1,2,1,2]})` and `Product(R[16]{[1,2,1,2,1,2]})`, in GAP3 notation, as we can see below:

```
gap> R:=Representations(H_4);;
gap> Product(R[15]{[1,2,1,2,1,2]});
[ [ u_1^3/2u_2^3/2u_3^3/2u_4^3/2, 0, 0, 0 ],
[ 0, u_1^3/2u_2^3/2u_3^3/2u_4^3/2, 0, 0 ],
[ 0, 0, u_1^3/2u_2^3/2u_3^3/2u_4^3/2, 0 ],
[ 0, 0, 0, u_1^3/2u_2^3/2u_3^3/2u_4^3/2] ]
gap> Product(R[16]{[1,2,1,2,1,2]});
[ [ -u_1^3/2u_2^3/2u_3^3/2u_4^3/2, 0, 0, 0 ],
[ 0, -u_1^3/2u_2^3/2u_3^3/2u_4^3/2, 0, 0 ],
[ 0, 0, -u_1^3/2u_2^3/2u_3^3/2u_4^3/2, 0 ],
[ 0, 0, 0, -u_1^3/2u_2^3/2u_3^3/2u_4^3/2 ] ]
```

We have $\theta(\omega_{4,5}(z_0)) = -\theta(\omega_{4,3}(z_0))$, which means that M is of defect zero i.e. $\theta(s_{\phi_{4,i}}) \neq 0$, where $i = 3$ or 5. The Schur elements $s_{\phi_{4,i}}$ have been determined in [11] §5.10, hence the condition $\theta(s_{\phi_{4,i}}) \neq 0$ becomes:

$$\frac{-2r\prod_{p=1}^{4}(r+\lambda_p^2)\prod_{r,l}(r+\lambda_r\lambda_l+\lambda_s\lambda_t)}{(\lambda_1\lambda_2\lambda_3\lambda_4)^4} \neq 0, \text{where } \{r,l,s,t\} = \{1,2,3,4\} \tag{6}$$

Therefore, an irreducible representation of B_3 of dimension 4 can be described by the explicit matrices A and B depending only on a choice of λ_1, λ_2, λ_3, λ_4 and a square root of $\lambda_1\lambda_2\lambda_3\lambda_4$ such that (6) is satisfied.

- $\underline{k=5}$: In this case, compared to the previous ones, we have two possibilities for S. The reason is that we have characters of dimension 5 and dimension 6, as well. Therefore, by Remark 5.1(i) and (ii) and (4) we either have $d_\theta([M]) = [S]$, where M is one irreducible $\mathbb{C}(\mathbf{v})\tilde{H}_5$-module of dimension 5 or $d_\theta([N]) = [S] + d_\theta([N'])$, where N, N' are some irreducible $\mathbb{C}(\mathbf{v})\tilde{H}_5$-modules of dimension 6 and 1, respectively.
In order to exclude the latter case, it is enough to show that N and N' are not in the same block. Therefore, at this point, we may assume that $\theta(\omega_\chi(z_0)) \neq \theta(\omega_\psi(z_0))$, for every irreducible character χ, ψ of W_5 of dimension 6 and 1, respectively. We

use GAP3 in order to calculate the central characters, as we did in the case where $k = 4$ and we have: $\theta(\omega_\psi(z_0)) = \lambda_i^6$, $i \in \{1,...,5\}$ and $\theta(\omega_\chi(z_0)) = -x^2yztw$, where $\{x,y,z,t,w\} = \{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5\}$. We notice that $\theta(\omega_\chi(z_0)) = -\lambda_j \det A$, $j \in \{1,...,5\}$. Therefore, the assumption $\theta(\omega_\chi(z_0)) \neq \theta(\omega_\psi(z_0))$ becomes $\det A \neq -\lambda_i^6\lambda_j^{-1}$, $i,j \in \{1,2,3,4,5\}$, where i, j are not necessarily distinct.

By this assumption we have that $d_\theta([M]) = [S]$, where M is some irreducible $\mathbb{C}(\mathbf{v})\tilde{H}_5$-module of dimension 5. We have again explicit matrix models for these representation (see [13] or the CHEVIE package of GAP3), therefore we can determine the matrices A and B. We notice that these matrices depend only on the choice of eigenvalues and of a fifth root of $\det A$.

Since $d_\theta([M])$ is irreducible either M is of defect 0 or it is in the same block with another irreducible module of dimension 5. However, since the central characters of the irreducible modules of dimension 5 are distinct fifth roots of $(u_1u_2u_3u_4u_5)^6$, we can exclude the latter case. Hence, M is of defect zero i.e. $\theta(s_\phi) \neq 0$, where ϕ is the character that corresponds to M. The Schur elements have been determined in [11] (see also Appendix A.3 in [5]) and one can also find them in CHEVIE package of GAP3; they are

$$\frac{5\prod_{i=1}^{5}(r+u_i)(r-\zeta_3u_i)(r-\zeta_3^2u_i)\prod_{i\neq j}(r^2+u_iu_j)}{(u_1u_2u_3u_4u_5)^7},$$

where r is a 5th root of $u_1u_2u_3u_4u_5$. However, due to the assumption $\det A \neq -\lambda_i^5$, $i \in \{1,2,3,4,5\}$ (case where $i = j$), we have that $\theta(r) + \lambda_i \neq 0$. Therefore, the condition $\theta(s_\phi) \neq 0$ becomes

$$\prod_{i=1}^{5}(\tilde{r}^2+\lambda_i\tilde{r}+\lambda_i^2)\prod_{i\neq j}(\tilde{r}^2+\lambda_i\lambda_j) \neq 0, \tag{7}$$

where $\tilde{r}$ is a fifth root of $\det A$.

To sum up, an irreducible representation of B_3 of dimension 5 can be described by the explicit matrices A and B, that one can find for example in CHEVIE package of GAP3, depending only on a choice of λ_1, λ_2, λ_3, λ_4, λ_5 and a fifth root of $\det A$ such that (7) is satisfied.

Remark 5.2.

1. We can generalize our results for a representation of B_3 over a field of positive characteristic, using similar arguments. However, the cases where $k = 4$ and $k = 5$ need some extra analysis; For $k = 4$ we have two irreducible $\mathbb{C}(\mathbf{v})\tilde{H}_4$-modules of dimension 4, which are not in the same block if we are in any characteristic but 2. However, when we are in characteristic 2, these two modules coincide and, therefore,

we obtain an irreducible module of B_3 which is of defect 0, hence we arrive to the same result as in characteristic 0. We have exactly the same argument for the case where $k = 5$ and we are over a field of characteristic 5.

2. The irreducible representations of B_3 of dimension at most 5 have been classified in [19]. Using a new framework, we arrived to the same results. The matrices A and B described by Tuba and Wenzl are the same (up to equivalence) with the matrices we provide in this paper. For example, in the case where $k = 3$, we have given explicit matrices A and B. If we take the matrices DAD^{-1} and DBD^{-1}, where D is the invertible matrix

$$D = \begin{bmatrix} -\lambda_1\lambda_2 - \lambda_3^2 & \lambda_1(\lambda_3 - \lambda_1) & (\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1) \\ (\lambda_2 - \lambda_1)(\lambda_3^2 + \lambda_1\lambda_2) & \lambda_1(2\lambda_2\lambda_1 - \lambda_1^2 + 2\lambda_1\lambda_3 - \lambda_3\lambda_2) & (\lambda_1 - \lambda_3)(\lambda_2^2 + \lambda_1\lambda_3) \\ 0 & \lambda_1(\lambda_1 - \lambda_3) & -\lambda_3\lambda_1(\lambda_1 + \lambda_2) \end{bmatrix},$$

we just obtain the matrices determined in [19]. (The matrix D is invertible since $\det D = \lambda_1(\lambda_1^2 + \lambda_2\lambda_3)(\lambda_3^2 + \lambda_1\lambda_2)^2 \neq 0$, due to (4).)

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