

The exceptional group G_{13}

1 Presentation

Let $\mathcal{R} := \mathbb{Z}[a, b^{\pm 1}, c, d^{\pm 1}]$. The Hecke algebra associated with G_{13} admits the following presentation:

$$\mathcal{H}_{G_{13}} = \langle s, t, u \mid ust = tust, \quad stust = ustus \quad s^2 = as + b, \quad t^2 = ct + d, \quad u^2 = cu + d \rangle.$$

2 Relations

Braid relations: $ustu = tust, \quad stust = ustus$.

Positive Hecke relations: $s^2 = as + b, \quad t^2 = ct + d, \quad u^2 = eu + f$.

Inverse Hecke relations: $s^{-1} = b^{-1}s - ab^{-1}, \quad t^{-1} = d^{-1}t - cd^{-1}, \quad u^{-1} = f^{-1}u - ef^{-1}$.

3 The basis \mathcal{B}_{13} and the spanning set \mathcal{C}_{13}

Let z be the central element $(stu)^3$. We recall that $|Z(G_{13})| = 4$. The basis \mathcal{B}_{13} consisting of 96 elements is the following:

$b_1 = 1$	$b_{25} = z$	$b_{49} = z^2$	$b_{73} = z^3$
$b_2 = u$	$b_{26} = zu$	$b_{50} = z^2u$	$b_{74} = z^3u$
$b_3 = s$	$b_{27} = zs$	$b_{51} = z^2s$	$b_{75} = z^3s$
$b_4 = ts$	$b_{28} = zts$	$b_{52} = z^2ts$	$b_{76} = z^3ts$
$b_5 = su$	$b_{29} = zsu$	$b_{53} = z^2su$	$b_{77} = z^3su$
$b_6 = us$	$b_{30} = zus$	$b_{54} = z^2us$	$b_{78} = z^3us$
$b_7 = tu$	$b_{31} = ztu$	$b_{55} = z^2tu$	$b_{79} = z^3tu$
$b_8 = tsu$	$b_{32} = ztsu$	$b_{56} = z^2tsu$	$b_{80} = z^3tsu$
$b_9 = tus$	$b_{33} = ztus$	$b_{57} = z^2tus$	$b_{81} = z^3tus$
$b_{10} = sts$	$b_{34} = zsts$	$b_{58} = z^2sts$	$b_{82} = z^3sts$
$b_{11} = stu$	$b_{35} = zstu$	$b_{59} = z^2stu$	$b_{83} = z^3stu$
$b_{12} = uts$	$b_{36} = zuts$	$b_{60} = z^2uts$	$b_{84} = z^3uts$
$b_{13} = t$	$b_{37} = zt$	$b_{61} = z^2t$	$b_{85} = z^3t$
$b_{14} = ut$	$b_{38} = zut$	$b_{62} = z^2ut$	$b_{86} = z^3ut$
$b_{15} = st$	$b_{39} = zst$	$b_{63} = z^2st$	$b_{87} = z^3st$
$b_{16} = tst$	$b_{40} = ztst$	$b_{64} = z^2tst$	$b_{88} = z^3tst$
$b_{17} = sut$	$b_{41} = zsut$	$b_{65} = z^2sut$	$b_{89} = z^3sut$
$b_{18} = ust$	$b_{42} = zust$	$b_{66} = z^2ust$	$b_{90} = z^3ust$
$b_{19} = tut$	$b_{43} = ztut$	$b_{67} = z^2tut$	$b_{91} = z^3tut$
$b_{20} = tsut$	$b_{44} = ztsut$	$b_{68} = z^2tsut$	$b_{92} = z^3tsut$
$b_{21} = tust$	$b_{45} = ztust$	$b_{69} = z^2tust$	$b_{93} = z^3tust$
$b_{22} = stst$	$b_{46} = zstst$	$b_{70} = z^2stst$	$b_{94} = z^3stst$
$b_{23} = stut$	$b_{47} = zstut$	$b_{71} = z^2stut$	$b_{95} = z^3stut$
$b_{24} = utst$	$b_{48} = zutst$	$b_{72} = z^2utst$	$b_{96} = z^3utst$

We expand the basis \mathcal{B}_{13} to a spanning set \mathcal{C}_{13} including the following four extra elements: $b_{97} := ttsu$, $b_{98} := tstsut$, $b_{99} := zututs$, and $b_{100} := zututst$.

4 The special cases

These are some equalities computed by hand which express a given $b_i b_j$ as a sum of other elements in $\mathcal{H}(G_{13})$.

Case 1: $s \cdot z^k \cdot \text{monomial}$ with $k \in \{1, 2, 3\}$.

$$s \cdot z^k \cdot \text{monomial} = z^k \cdot s \cdot \text{monomial}.$$

Case 2: $t \cdot z^k \cdot \text{monomial}$ with $k \in \{1, 2, 3\}$.

$$t \cdot z^k \cdot \text{monomial} = z^k \cdot t \cdot \text{monomial}.$$

Case 3: $u \cdot z^k \cdot \text{monomial}$ with $k \in \{1, 2, 3\}$.

$$u \cdot z^k \cdot \text{monomial} = z^k \cdot u \cdot \text{monomial}.$$

Case 4: $z^k stsut^l$ with $k \in \{1, 2, 3\}$ and $l \in \{0, 1\}$.

$$z^k stsut^l = a z^k tsut^l + bc z^k ut^l + bda z^k s^{-1} t^{-1} ut^l + b^2 dc z^k s^{-1} t^{-1} s^{-1} t^l + b^2 d^2 z^{k-1} tust^{l+2}.$$

Case 5: $z^k stust^l$ with $k \in \{0, 1, 2\}$ and $l \in \{0, 1\}$.

$$z^k stust^l = z^{k+1} t^{-1} s^{-1} u^{-1} t^{l-2}.$$

Case 6: $z^k ustst^l$ with $k \in \{0, 1, 2\}$ and $l \in \{0, 1\}$.

$$z^k ustst^l = z^{k+1} t^{-2} s^{-1} u^{-1} t^{l-1}.$$

Case 7: $z^k tstst^l$ with $k \in \{0, 1\}$ and $l \in \{0, 1\}$.

$$\begin{aligned} z^k tstst^l &= (-cd^{-1}) z^{k+1} t^{-1} s^{-1} u^{-1} t^{l-1} + (-d^{-2}c) z^{k+1} s^{-1} u^{-1} t^l + (-d^{-2}ab^{-1}) z^{k+1} t^l + \\ &\quad + (-d^{-3}b^{-1}c) z^{k+1} ust^l + (-d^{-4}b^{-1}) z^{k+2} s^{-1} t^{-1} s^{-1} t^{l-1} + d^{-4}b^{-1}c z^{k+1} tust^{l+1}. \end{aligned}$$

Case 8: $z^k tstst^l$ with $k \in \{2, 3\}$ and $l \in \{0, 1\}$.

$$z^k tstst^l = c z^k stst^l + da z^k st^l + dbc z^k t^{l-1} + d^2 ba z^k t^{-1} s^{-1} t^{l-1} + d^2 b^2 z^{k-1} ust^2 ut^l.$$

Case 9: $z^k tstut^l$ with $k \in \{0, 1\}$ and $l \in \{0, 1\}$.

$$z^k tstut^l = z^{k+1} s^{-1} u^{-1} t^{-2} s^{-1} t^l.$$

Case 10: $z^k tstut^l$ with $k \in \{2, 3\}$ and $l \in \{0, 1\}$.

$$\begin{aligned} z^k tstut^l &= c z^k stut^l + da z^k ut^l + dbc z^k t^{-1} s^{-1} ut^l + d^2 bc z^k t^{-1} s^{-1} t^{l-1} + d^2 bc z^k u^{-1} t^{-1} s^{-1} t^l + \\ &\quad + d^4 ba z^{k-1} ust^l + (-d^2 bc^2) z^k t^{-1} s^{-1} u^{-1} t^{l-1} + d^4 b^2 c z^{k-1} t^l + d^5 b^2 a z^{k-1} s^{-1} u^{-1} t^l + \\ &\quad + d^5 b^3 z^{k-2} utu^2 st^{l+1}. \end{aligned}$$

Case 11: $z^k sutst^l$ with $k \in \{1, 2, 3\}$ and $l \in \{0, 1\}$.

$$z^k sutst^l = a z^k utst^l + bc z^k s^{-1} tst^l + bdc z^k s^{-1} u^{-1} st^l + bd^2 a z^k s^{-1} u^{-1} t^{l-1} + b^2 d^2 z^{k-1} t^2 ust^{l+1}.$$

Case 12: $z^k ustut^l$ with $k \in \{0, 1, 2, 3\}$ and $l \in \{0, 1\}$.

$$z^k ustut^l = z^k tust^{l+1}.$$

Case 13: $z^k utsut^l$ with $k \in \{0, 1, 2, 3\}$ and $l \in \{0, 1\}$.

$$z^k utsut^l = c z^k tsut^l + cd z^k u^{-1} sut^l + d^2 a z^k u^{-1} t^{-1} ut^l + d^2 bc z^k u^{-1} t^{-1} s^{-1} t^l + d^3 b z^k t^{-1} s^{-1} u^{-1} t^{l-1}.$$

Case 14: $z^k usut^l$ with $k \in \{1, 2, 3\}$ and $l \in \{0, 1\}$.

$$z^k usut^l = c z^k ust^l + da z^k t^l + dbc z^k s^{-1} u^{-1} t^l + d^2 b z^{k-1} stu^2 st^{l+1}.$$

Case 15: $z^k sust^l$ with $k \in \{0, 1, 2\}$ and $l \in \{0, 1\}$.

$$z^k sust^l = d^{-1} z^{k+1} u^{-1} t^{-1} s^{-1} t^{l-1} + (-cd^{-1}) z^{k+1} t^{-1} s^{-1} u^{-1} t^{l-2}.$$

Case 16: $z^3 sust^l$ with $l \in \{0, 1\}$.

$$z^3 sust^l = a z^3 ust^l + bc z^3 t^l + abd z^3 s^{-1} u^{-1} t^l + b^2 d^2 z^2 tstut^l + b^2 dc z^2 t^2 ust^{l+1}.$$

Case 17: $z^k utust^l$ with $k \in \{0, 1, 2, 3\}$ and $l \in \{0, 1\}$.

$$z^k utust^l = z^k u^2 stut^{l-1}.$$

Case 18: $usut^l$ with $l \in \{0, 1\}$.

$$usut^l = d^{-1} z t^{-1} s^{-1} t^{-1} s^{-1} t^l + (-cd^{-1}) tust^{l+1}.$$

Case 19: $z^3 ustst^l$ with $l \in \{0, 1\}$.

$$\begin{aligned} z^3 ustst^l = & c z^3 stst^l + ad z^3 u^{-1} tst^l + bdc z^3 u^{-1} t^l + bd^2 a z^3 u^{-1} s^{-1} t^{l-1} + b^2 d^2 c z^2 stu^2 t^l + \\ & + b^2 d^3 c z^2 sut^l + b^2 d^4 c z^2 u^{-1} t^{-1} sut^l + b^2 d^5 a z^2 u^{-1} t^{l-1} + b^3 d^5 c z^2 u^{-1} t^{-1} u^{-1} s^{-1} t^l + \\ & + b^3 d^6 a z ust^{l+1} + (-cab^3 d^6) zst^{l+1} + cb^4 d^7 zu^{-1} t^l + cb^3 d^6 zstst^l + b^4 d^8 st^2 us^2 t^{l+1}. \end{aligned}$$

Case 20: $z^k utut^l$ with $k \in \{0, 1\}$ and $l \in \{0, 1\}$.

$$\begin{aligned} z^k utut^l = & (-ab^{-1}) z^k ustut^l + (-b^{-1} d^{-2} c) z^{k+1} s^{-1} t^l + (-b^{-1} d^{-1} c) z^{k+1} t^{-1} s^{-1} u^{-1} t^l + \\ & + (-b^{-1} d^{-3} c) z^{k+1} usts^{-2} t^l + a^2 b^{-3} d^{-3} z^{k+1} ustut^l + (-ab^{-3} d^{-3}) z^{k+1} stust^{l+1} + \\ & + b^{-2} d^{-3} z^{k+2} t^{-1} u^{-1} t^{-1} s^{-2} t^l. \end{aligned}$$

Case 21: $z^k utut^l$ with $k \in \{2, 3\}$ and $l \in \{0, 1\}$.

$$\begin{aligned} z^k utut^l = & c z^k tut^l + cd z^k t^l + cd^2 z^k u^{-1} t^{l-1} + ad^3 z^k t^{-1} s^{-1} u^{-1} t^{l-1} + bd^3 c z^{k-1} ust^{l+1} + \\ & + bd^4 c z^{k-1} st^l + bd^5 c z^{k-1} s^2 t^{-1} s^{-1} u^{-1} t^l + abd^6 z^{k-1} st^{-1} s^{-1} u^{-1} t^{l-1} + b^2 d^6 z^{k-2} s^2 tut^{l+1}. \end{aligned}$$

Case 22: $z^k tutst^l$ with $k \in \{2, 3\}$ and $l \in \{0, 1\}$.

$$\begin{aligned} z^k tutst^l = & a z^k tut^{l+1} + bc z^k tus^{-1} t^l + bdc z^k s^{-1} t^l + cbd^2 z^k u^{-1} t^{-1} s^{-1} t^l + abd^3 z^{k-1} ustut^l + \\ & + cb^2 d^3 z^{k-1} u^2 t^l + b^2 d^4 c z^{k-1} ust^{-1} s^{-1} t^l + b^3 d^5 z^{k-2} utust^{l+2} + ab^2 d^5 z^{k-2} stust^{l+3}. \end{aligned}$$

Case 23: $z^k tutst^l$ with $k \in \{0, 1\}$ and $l \in \{0, 1\}$.

$$\begin{aligned} z^k tutst^l = & (-ab^{-1}) z^k stust^{l+1} + (-b^{-1} d^{-2} c) z^{k+1} t^l + (-b^{-2} d^{-3} c) z^{k+1} ustst^l + \\ & + (-a^2 b^{-2} d^{-2}) z^{k+1} s^{-1} t^{-1} ust^{l+1} + b^{-2} d^{-4} z^{k+2} s^{-1} t u^{-1} t^{l-1} + (-b^{-1} d^{-4} c) z^{k+1} s^{-1} t^2 ust^{l+1} + \\ & + (-ab^{-2} d^{-4} c) z^{k+2} s^{-2} u^{-1} t^{l-1} + (-ab^{-1} d^{-3}) z^{k+1} s^{-2} tust^{l+1} + (-a^2 b^{-2} d^{-3} c) z^{k+1} s^{-1} ust^{l+1} + \\ & + ab^{-2} d^{-3} c z^{k+2} s^{-1} t^{-2} s^{-1} u^{-1} t^{l-1} + (-b^{-1} c) z^k stus^2 t^l + (-d^{-1} c^2) z^k ustut^l + \\ & + (-ab^{-1} d^{-1} c^2) z^k stust^{l+1}. \end{aligned}$$

Case 24: $sutst^l$ with $l \in \{0, 1\}$.

$$sutst^l = a utst^l + bc s^{-1} tst^l + bdc s^{-1} u^{-1} st^l + bdtsts^{-1} u^{-1} t^{l-1} + (-bdc) tu^{-1} t^{l-1}.$$

Case 25: $stsut^l$ with $l \in \{0, 1\}$.

$$stsut^l = zu^{-1} t^{-1} u^{-1} t^{-1} s^{-1} t^l.$$