The exceptional group G_4

1 Presentation

Let $R := \mathbb{Z}[a, b, c^{\pm 1}]$. The Hecke algebra associated with G_4 admits the following presentation:

$$H_{G_4} = \langle s, t \mid sts = tst, \quad s^3 = as^2 + bs + c, \quad t^3 = at^2 + bt + c \rangle.$$

2 Relations

Braid relation : $sts = tst$.

Positive Hecke relations : $s^3 = as^2 + bs + c$ and $t^3 = at^2 + bt + c$

Inverse Hecke relations : $s^{-1} = c^{-1}s^2 - ac^{-1}s - bc^{-1}1$ and $t^{-1} = c^{-1}t^2 - ac^{-1}t - bc^{-1}1$.

3 The basis $\mathcal{B}_4$

b_1	$=$	1	b_{13}	$=$	st^2
b_2	$=$	s	b_{14}	$=$	st
b_3	$=$	s^2	b_{15}	$=$	st^2s
b_4	$=$	$ststst$	b_{16}	$=$	sts
b_5	$=$	$ststst s$	b_{17}	$=$	st^2s^2
b_6	$=$	$stststs^2$	b_{18}	$=$	sts^2
b_7	$=$	t^2	b_{19}	$=$	s^2t^2
b_8	$=$	t	b_{20}	$=$	s^2t
b_9	$=$	t^2s	b_{21}	$=$	s^2t^2s
b_{10}	$=$	ts	b_{22}	$=$	s^2ts
b_{11}	$=$	t^2s^2	b_{23}	$=$	$s^2t^2s^2$
b_{12}	$=$	ts^2	b_{24}	$=$	s^2ts^2

4 Special cases

We denote by $eq_t(m)$, where m is an element of the image of the corresponding braid group inside H_{G4}, an element m' which can transform into m by applying the braid relation and which ends with t.

Case 1 : sts^mt with $m \in \{2, 3\}$.

$$sts^mt = \mathrm{eq}_t(sts^mts)s^{-1}.$$

Case 2 : An element that contains 3 cubes in a row.

We give an example here. Let $s^2t^2s^2t^2$. We write $s^2t^2s^2t^2 = (s^2t^2s^2ts) \cdot s^{-1}t$. We apply now the braid relation to the element $s^2t^2s^2ts$ until we transform it to an element that ends with t and contains a cube. In this example we have: $(s^2t^2s^2ts) \cdot s^{-1}t = s^2t^3st^2s^{-1}t$. We now apply the positive Hecke relation and we have: $s^2t^3st^2s^{-1}t = as^2t^2st^2s^{-1}t + bs^2tst^2s^{-1}t + cs^3t^2s^{-1}t$. We now check if by applying the braid relation, we can omit the negative power. We have: $s^2t^2st^2s^{-1}t = ts^2t^2$ and $s^2tst^2s^{-1}t = s^4t^2$. If this is not the case, we apply the inverse Hecke relation.

Case 3 : ts^2t^2x with x an element of the image of the corresponding braid group inside H_{G4}.

$$ts^2t^2x = s^{-1}t^2st^3x.$$

Case 4 : st^2s^2tx with x an element of the image of the corresponding braid group inside H_{G4}.

$$st^2s^2tx = \mathrm{eq}_t(st^2s^2ts)s^{-1}x.$$

Case 5 : s^2t^2x and t^2s^2x with x an element of the image of the corresponding braid group inside H_{G4}.

We write for example, $s^2t^2x = s^2t^2x \cdot s \cdot s^{-1}$ or $s^2t^2x = s^2t^2x \cdot t \cdot t^{-1}$, depending on x.

5 The extra condition

Let z be the central element $ststst$. We recall that $|Z(G_4)| = 2$. In order to prove the extra condition, we will write $\tau(z^2b_i^{-1})$ for all $b_i \in \mathcal{B}_4 \setminus \{1\}$ as $\mathbb{Z}[a,b,c^{\pm 1}]$-linear combinations of elements of the form $\tau(b_jb_l)$ with $b_j, b_l \in \mathcal{B}_8$ and elements of the form $\tau(z^2b_r^{-1})$ with $r \neq i$ that have already been calculated. Therefore, we can show that $\tau(z^4b_i^{-1}) = 0$ for all $b_i \in \mathcal{B}_4 \setminus \{1\}$ using the entries of the matrix A, which are computed by the `C++` program.

$$\begin{aligned}
\tau(z^2b_2^{-1}) &= \tau(z^2s^{-1}) = c^{-1}\tau(z^2s^2) - ca^{-1}\tau(z^2s) - cb^{-1}\tau(z^2) = c^{-1}\tau(b_4b_6) - ca^{-1}\tau(b_4b_5) - cb^{-1}\tau(b_4b_4) = \\
&= c^{-1}(a^2c^4 + bc^4) - ca^{-1}(ac^4) - cb^{-1}(c^4) = 0. \\
\tau(z^2b_3^{-1}) &= \tau(zts^2t) = \tau(zs^2t^2) = \tau(b_6b_7) = 0. \\
\tau(z^2b_4^{-1}) &= \tau(z) = \tau(b_4) = 0. \\
\tau(z^2b_5^{-1}) &= \tau(t^2st^2) = \tau(b_7b_{13}) = 0. \\
\tau(z^2b_6^{-1}) &= \tau(ts^2t) = \tau(b_{12}b_8) = 0. \\
\tau(z^2b_7^{-1}) &= \tau(zst^2s) = \tau(b_4b_{15}) = 0. \\
\tau(z^2b_8^{-1}) &= \tau(zs^2ts^2) = \tau(b_4b_{24}) = 0. \\
\tau(z^2b_9^{-1}) &= \tau(zt^2s) = \tau(b_4b_9) = 0. \\
\tau(z^2b_{10}^{-1}) &= \tau(zsts^2) = \tau(b_4b_{18}) = 0. \\
\tau(z^2b_{11}^{-1}) &= \tau(zs^{-1}t^2s) = \tau(zt^2) = \tau(b_4b_7) = 0. \\
\tau(z^2b_{12}^{-1}) &= \tau(zts^2) = \tau(b_4b_{12}) = 0. \\
\tau(z^2b_{13}^{-1}) &= \tau(z^2t^{-2}s^{-1}) = \tau(z^2s^{-1}t^{-2}) = \tau(z^2b_9^{-1}) = 0. \\
\tau(z^2b_{14}^{-1}) &= \tau(z^2t^{-1}s^{-1}) = \tau(z^2s^{-1}t^{-1}) = \tau(z^2b_{10}^{-1}) = 0. \\
\tau(z^2b_{15}^{-1}) &= \tau(z^2s^{-1}t^{-2}s^{-1}) = \tau(z^2s^{-2}t^{-2}) = \tau(z^2b_{11}^{-1}) = 0. \\
\tau(z^2b_{16}^{-1}) &= \tau(z^2s^{-1}t^{-1}s^{-1}) = \tau(z^2s^{-2}t^{-1}) = \tau(z^2b_{12}^{-1}) = 0. \\
\tau(z^2b_{17}^{-1}) &= \tau(sts^4t) = \tau(b_{18}b_{20}) = 0. \\
\tau(z^2b_{18}^{-1}) &= \tau(t^2s^2ts) = \tau(b_7b_{22}) = 0. \\
\tau(z^2b_{19}^{-1}) &= \tau(z^2t^{-2}s^{-2}) = \tau(z^2s^{-2}t^{-2}) = \tau(z^2b_{11}^{-1}) = 0. \\
\tau(z^2b_{20}^{-1}) &= \tau(z^2t^{-1}s^{-2}) = \tau(z^2s^{-2}t^{-1}) = \tau(z^2b_{12}^{-1}) = 0. \\
\tau(z^2b_{21}^{-1}) &= \tau(z^2s^{-1}t^{-2}s^{-2}) = \tau(z^2s^{-2}t^{-2}s^{-1}) = \tau(z^2b_{17}^{-1}) = 0. \\
\tau(z^2b_{22}^{-1}) &= \tau(z^2s^{-1}t^{-1}s^{-2}) = \tau(z^2s^{-2}t^{-1}s^{-1}) = \tau(z^2b_{18}^{-1}) = 0. \\
\tau(z^2b_{23}^{-1}) &= \tau(zs^{-1}t^2s^{-1}) = \tau(zs^{-2}t^2) = \tau(ts^2t^3) = \tau(t^2s^2t^2) = \tau(b_{11}b_7) = 0. \\
\tau(z^2b_{24}^{-1}) &= \tau(zt) = \tau(b_4b_8) = 0.
\end{aligned}$$