The exceptional group G_5

1 Presentation

Let $R := \mathbb{Z}[a, b, c^{\pm 1}, d, e, f^{\pm 1}]$. The Hecke algebra associated with G_5 admits the following presentation:

$$H_{G_5} = \langle s, t \mid stst = tsts, \quad s^3 = as^2 + bs + c, \quad t^3 = dt^2 + et + f \rangle.$$

2 Relations

Braid relation : $stst = tsts$.

Positive Hecke relations : $s^3 = as^2 + bs + c, \quad t^3 = dt^2 + et + f$.

Inverse Hecke relations : $s^{-1} = c^{-1}s^2 - ac^{-1}s - bc^{-1}, \quad t^{-1} = f^{-1}t^2 - df^{-1}t - ef^{-1}$.

3 The basis $\mathcal{B}_5$

Let z be the central element $stst$. We recall that $|Z(G_5)| = 6$. We also set

$$\mathcal{E}_5 = \left\{1, s, s^2, t, t^2, st, s^2t, st^2, s^2t^2, t^{-1}s, t^{-1}st, t^{-1}st^2\right\}.$$

The basis $\mathcal{B}_5$ consisting of 72 elements is the following:

$b_1 = 1$	$b_{13} = z$	$b_{25} = z^2$	$b_{37} = z^3$	$b_{49} = z^4$	$b_{61} = z^5$
$b_2 = s$	$b_{14} = zs$	$b_{26} = z^2s$	$b_{38} = z^3s$	$b_{50} = z^4s$	$b_{62} = z^5s$
$b_3 = s^2$	$b_{15} = zs^2$	$b_{27} = z^2s^2$	$b_{39} = z^3s^2$	$b_{51} = z^4s^2$	$b_{63} = z^5s^2$
$b_4 = t$	$b_{16} = zt$	$b_{28} = z^2t$	$b_{40} = z^3t$	$b_{52} = z^4t$	$b_{64} = z^5t$
$b_5 = t^2$	$b_{17} = zt^2$	$b_{29} = z^2t^2$	$b_{41} = z^3t^2$	$b_{53} = z^4t^2$	$b_{65} = z^5t^2$
$b_6 = st$	$b_{18} = zst$	$b_{30} = z^2st$	$b_{42} = z^3st$	$b_{54} = z^4st$	$b_{66} = z^5st$
$b_7 = s^2t$	$b_{19} = zs^2t$	$b_{31} = z^2s^2t$	$b_{43} = z^3s^2t$	$b_{55} = z^4s^2t$	$b_{67} = z^5s^2t$
$b_8 = st^2$	$b_{20} = zst^2$	$b_{32} = z^2st^2$	$b_{44} = z^3st^2$	$b_{56} = z^4st^2$	$b_{68} = z^5st^2$
$b_9 = s^2t^2$	$b_{21} = zs^2t^2$	$b_{33} = z^2s^2t^2$	$b_{45} = z^3s^2t^2$	$b_{57} = z^4s^2t^2$	$b_{69} = z^5s^2t^2$
$b_{10} = t^{-1}s$	$b_{22} = zt^{-1}s$	$b_{34} = z^2t^{-1}s$	$b_{46} = z^3t^{-1}s$	$b_{58} = z^4t^{-1}s$	$b_{70} = z^5t^{-1}s$
$b_{11} = t^{-1}st$	$b_{23} = zt^{-1}st$	$b_{35} = z^2t^{-1}st$	$b_{47} = z^3t^{-1}st$	$b_{59} = z^4t^{-1}st$	$b_{71} = z^5t^{-1}st$
$b_{12} = t^{-1}st^2$	$b_{24} = zt^{-1}st^2$	$b_{36} = z^2t^{-1}st^2$	$b_{48} = z^3t^{-1}st^2$	$b_{60} = z^4t^{-1}st^2$	$b_{72} = z^5t^{-1}st^2$

4 Special cases

Case 1 : $s^{m_1}t^{n_1}z^ks^{m_2}t^{n_2}$, $k \in \{1, \dots, 5\}$, $m_1, n_1 \in \{0, 1\}$ and $m_2, n_2 \in \{0, 1, 2\}$.

$$s^{m_1}t^{n_1}z^ks^{m_2}t^{n_2} = z^ks^{m_1}t^{n_1}s^{m_2}t^{n_2}.$$

Case 2 : $s^{m_1}t^{n_1}z^kt^{-1}st^{n_2}$, $k \in \{1, \dots, 5\}$, $m_1, n_1 \in \{0, 1\}$ and $n_2 \in \{0, 1, 2\}$.

$$s^{m_1}t^{n_1}z^kt^{-1}st^{n_2} = z^ks^{m_1}t^{n_1-1}st^{n_2}.$$

Case 3 : $z^k s^p t^2 s t^m$, $k \in \{0,\dots,5\}$, $p \in \{0,1\}$, $m \in \{0,1,2\}$.

$$z^k s^p t^2 s t^m = d z^k s^p t s t^m + e z^k s^{p+1} t^m + f z^k s^p t^{-1} s t^m.$$

Case 4 : $z^k t^m s^2 t^n$, $k \in \{1,\dots,5\}$, $m \in \{1,2\}$, $n \in \{0,1,2\}$.

$$z^k t^m s^2 t^n = a z^k t^m s t^n + b z^k t^{m+n} + c z^{k-1} t^{m+1} s t^{n+1}.$$

Case 5 : $z^k s^p t s^m t^n$, $k \in \{0,\dots,4\}$, $p \in \{0,1,2\}$, $m \in \{1,2\}$ and $n \in \{0,1,2\}$.

$$z^k s^p t s^m t^n = z^{k+1} s^{p-1} t^{-1} s^{m-1} t^n.$$

Case 6 : $z^k s^p t^{-1} s t^n$, $k \in \{0,\dots,3\}$, $p \in \{1,2\}$, $n \in \{0,1,2\}$.

$$z^k s^p t^{-1} s t^n = f^{-1} z^{k+1} s^p t s^{-1} t^{n-1} + (-f^{-1} d) z^k s^p t s t^n + (-f^{-1} e) z^k s^{p+1} t^n.$$

Case 7 : $z^k s t^{-1} s t^n$, $k \in \{4,5\}$, $n \in \{0,1,2\}$.

$$\begin{aligned} z^k s t^{-1} s t^n = {} & a z^k t^{-1} s t^n + b z^{k-1} t s^2 t^n + ac z^{k-1} s^{-1} t s t^n + bc z^{k-1} s^{-1} t^{n+1} + c^2 d z^{k-1} s^{-2} t^n + \\ & + c^2 e z^{k-2} t^{n+1} + c^2 f z^{k-3} t s^2 t^{n+1}. \end{aligned}$$

Case 8 : $z^5 s^m t s t^n$, $m \in \{0,1\}$, $n \in \{0,1,2\}$.

$$\begin{aligned} z^5 s^m t s t^n = {} & d z^5 s^{m+1} t^n + e z^5 s^m t^{-1} s t^n + af z^5 s^m t^{n-2} + bf z^4 s^m t^{-1} s t^{n+1} + acf z^4 s^{m-1} t^n + \\ & + bcfd z^3 s^{m+1} t s^{-1} t^n + bcfe z^3 s^m t^n + bcf^2 z^2 s^{m+2} t^{n+1} + c^2 fe z^2 s^{m+1} t^{n+2} + \\ & + cfd z^4 s^m t^{n-1} + (-cafd) z^3 s^{m+1} t^{n+1} + (-bfd) z^4 s^m t^{-1} s t^n + abfd z^4 s^m t^{n-1} + \\ & + b^2 fd z^3 s^{m+1} t^{n+1} + c^2 f^2 d z s^{m+1} t s^2 t^{n+1} + c^2 f^2 e z s^{m+3} t^{n+1} + ac^2 f^3 s^{m+2} t s^2 t^{n+1} + \\ & + bc^2 f^3 z s^{m+1} t^n + c^3 f^3 s^{m+2} t^{n+2}. \end{aligned}$$

5 The extra condition

In order to prove the extra condition, we will write $\tau(z^6 b_i^{-1})$ for all $b_i \in \mathcal{B}_5 \setminus \{1\}$ as $\mathbb{Z}[a,b,c^{\pm 1},d,e,f^{\pm 1}]$-linear combinations of elements of the form $\tau(b_j b_l)$ with $b_j, b_l \in \mathcal{B}_5$. Therefore, we can show that $\tau(z^6 b_i^{-1}) = 0$ for all $b_i \in \mathcal{B}_5 \setminus \{1\}$ using the entries of the matrix A, which are computed by the `C++`/SAGE program.

Let $b_i \in \mathcal{B}_5 \setminus \{1\}$. We first consider the case where $i > 12$. We write b_i as $z^k b_m$, for $k \in \{1,\dots,5\}$ and $b_m \in \mathcal{E}_5$. Since τ is a trace function, we have that $\tau(z^{6-k} b_m^{-1}) = \tau(z^{6-k} s^{p_1} t^{p_2})$, where $p_1 \in \{-2,-1,0\}$ and $p_2 \in \{-2,-1,0,1\}$. Using now the inverse Hecke relations, we can write $\tau(z^{6-k} s^{p_1} t^{p_2})$ as a $\mathbb{Z}[a,b,c^{\pm 1},d,e,f^{\pm 1}]$-linear combination of elements of the form $\tau(z^{6-k} s^{q_1} t^{q_2})$, with $q_1, q_2 \in \{0,1,2\}$. Since $k \in \{1,\dots,5\}$, we have that $z^{6-k} s^{q_1} t^{q_2} \in \mathcal{B}_5 \setminus \{1\}$ and, hence, $\tau(z^{6-k} b_m^{-1}) = 0$. We now consider the case where $i \leq 12$. We have:

$$\begin{aligned}
\tau(z^6 b_2^{-1}) &= \tau(z^5 tst) = \tau(z^5 st^2) = \tau(b_{68}) = 0. \\
\tau(z^6 b_3^{-1}) &= \tau(z^4 tst^2 st) = \tau(z^4 st^2 st^2) = \tau(b_{56} b_8) = 0. \\
\tau(z^6 b_4^{-1}) &= \tau(z^5 sts) = \tau(z^5 s^2 t) = \tau(b_{67}) = 0. \\
\tau(z^6 b_5^{-1}) &= \tau(z^4 sts^2 ts) = \tau(z^4 s^2 ts^2 t) = \tau(b_{55} b_7) = 0. \\
\tau(z^6 b_6^{-1}) &= \tau(z^5 st) = \tau(b_{66}) = 0. \\
\tau(z^6 b_7^{-1}) &= \tau(z^5 sts^{-1}) = \tau(z^5 t) = \tau(b_{64}) = 0. \\
\tau(z^6 b_8^{-1}) &= \tau(z^5 tst^{-1}) = \tau(z^5 s) = \tau(b_{62}) = 0. \\
\tau(z^6 b_9^{-1}) &= \tau(z^5 stst^{-1} s^{-2}) = \tau(z^5 tst^{-1} s^{-1}) = \tau(z^4 ts^2 t) = \tau(z^4 s^2 t^2) = \tau(b_{57}) = 0. \\
\tau(z^6 b_{10}^{-1}) &= \tau(z^5 tst^2) = \tau(z^5 st^3) = \tau(b_{68} b_4) = 0. \\
\tau(z^6 b_{11}^{-1}) &= \tau(z^5 st^2) = \tau(b_{68}) = 0. \\
\tau(z^6 b_{12}^{-1}) &= \tau(z^5 stst^{-1} s^{-1} t) = \tau(z^5 tstst^{-1} s^{-1}) = \tau(z^5 st) = \tau(b_{66}) = 0.
\end{aligned}$$