The exceptional group G_6

1 Presentation

Let $R := \mathbb{Z}[a, b^{\pm 1}, c, d, e^{\pm 1}]$. The Hecke algebra associated with G_6 admits the following presentation:

$$H_{G_6} = \langle s, t \mid ststst = tststs, \quad s^2 = as + b, \quad t^3 = ct^2 + dt + e \rangle.$$

2 Relations

Braid relation : $ststst = tststs$.

Positive Hecke relations : $s^2 = as + b, \quad t^3 = ct^2 + dt + e$

Inverse Hecke relations : $s^{-1} = b^{-1}s - ab^{-1}, \quad t^{-1} = e^{-1}t^2 - ce^{-1}t - de^{-1}$

3 The basis $\mathcal{B}_6$

Let z be the central element $ststst$. We recall that $|Z(G_6)| = 4$. We also set

$$\mathcal{E}_6 = \left\{1, t, t^2, s, st, st^2, ts, t^2s, tst, tst^2, t^2st, t^2st^2\right\}.$$

The basis $\mathcal{B}_6$ consisting of 48 elements is the following:

$$\begin{array}{lllllllllll}
b_1 = 1 & & b_{13} = z & & b_{25} = z^2 & & b_{37} = z^3 \\
b_2 = t & & b_{14} = zt & & b_{26} = z^2t & & b_{38} = z^3t \\
b_3 = t^2 & & b_{15} = zt^2 & & b_{27} = z^2t^2 & & b_{39} = z^3t^2 \\
b_4 = s & & b_{16} = zs & & b_{28} = z^2s & & b_{40} = z^3s \\
b_5 = st & & b_{17} = zst & & b_{29} = z^2st & & b_{41} = z^3st \\
b_6 = st^2 & & b_{18} = zst^2 & & b_{30} = z^2st^2 & & b_{42} = z^3st^2 \\
b_7 = ts & & b_{19} = zts & & b_{31} = z^2ts & & b_{43} = z^3ts \\
b_8 = t^2s & & b_{20} = zt^2s & & b_{32} = z^2t^2s & & b_{44} = z^3t^2s \\
b_9 = tst & & b_{21} = ztst & & b_{33} = z^2tst & & b_{45} = z^3tst \\
b_{10} = tst^2 & & b_{22} = ztst^2 & & b_{34} = z^2tst^2 & & b_{46} = z^3tst^2 \\
b_{11} = t^2st & & b_{23} = zt^2st & & b_{35} = z^2t^2st & & b_{47} = z^3t^2st \\
b_{12} = t^2st^2 & & b_{24} = zt^2st^2 & & b_{36} = z^2t^2st^2 & & b_{48} = z^3t^2st^2
\end{array}$$

4 Special cases

Case 1 : $t^{m_1}s^{l_1}t^{n_1}z^kt^{m_2}s^{l_2}t^{n_2}$, $k \in \{1, 2, 3\}$, $l_1, l_2 \in \{0, 1\}$, $m_1, m_2, n_1, n_2 \in \{0, 1, 2\}$.

$$t^{m_1}s^{l_1}t^{n_1}z^kt^{m_2}s^{l_2}t^{n_2} = z^kt^{m_1}s^{l_1}t^{n_1+m_2}s^{l_2}t^{n_2}.$$

Case 2 : $z^kt^mstst^n$, $k \in \{0, 1, 2\}$, $m, n \in \{0, 1, 2\}$.

$$z^kt^mstst^n = z^{k+1}t^{m-1}s^{-1}t^{n-1}.$$

Case 3 : $z^k t^m st^2 st^n$, $k \in \{1,2,3\}$, $m,n \in \{0,1,2\}$.

$$z^k t^m st^2 st^n \quad = \quad cz^k t^m stst^n + dz^k t^m s^2 t^n + aez^k t^{m-1} st^n + abez^k t^m s^{-1} t^{n-1} + eb^2 z^{k-1} t^{m+1} st^{n+1}.$$

Case 4 : $z^3 t^m stst^n$, $m,n \in \{0,1,2\}$.

$$\begin{aligned} z^3 t^m stst^n &= az^3 t^{m+1} st^n + abz^3 t^m s^{-1} t^{n+1} + cb^2 z^3 t^m s^{-2} t^n + db^2 z^2 t^{m+1} st^{n+1} + ab^2 ez^2 t^{m+n+1} + \\ &\quad + eb^3 zt^{m+1} st^2 st^{n+1}. \end{aligned}$$

Case 5 : $t^m st^2 st^n$, $m,n \in \{0,1,2\}$.

$$\begin{aligned} t^m st^2 st^n &= -ab^{-2} zt^m stst^{n-2} + b^{-2} e^{-1} z^2 t^{m-1} s^{-1} tst^{n-1} + (-b^{-2} ce^{-1}) z^2 t^{m+n-2} + (-b^{-1} a) zt^{n+m-1} + \\ &\quad + (-b^{-2} de^{-1} a) zt^m stst^{n-1} + (-b^{-1} de^{-1}) zt^m st^n. \end{aligned}$$

Case 6 : The element z^4

$$\begin{aligned} z^4 &= az^3 tstst + bcz^3 tst + bdz^2 tsts^2 tst + abez^2 ts^2 tst + ab^2 ez^2 tst^2 + b^3 ecz^2 s^{-1} tst + b^5 e^2 t^2 st^3 st + \\ &\quad + b^3 edzts^2 t^2 st + ab^3 e^2 zst^2 st + ab^4 e^2 zts^{-1} tst. \end{aligned}$$

5 The extra condition

In order to prove the extra condition, we will write $\tau(z^4 b_i^{-1})$ for all $b_i \in \mathcal{B}_6 \setminus \{1\}$ as $\mathbb{Z}[a, b^{\pm 1}, c, d, e^{\pm 1}]$-linear combinations of elements of the form $\tau(b_j b_l)$ with $b_j, b_l \in \mathcal{B}_6$ and elements of the form $\tau(z^4 b_r^{-1})$ with $r \neq i$ that have already been calculated. Therefore, we can show that $\tau(z^4 b_i^{-1}) = 0$ for all $b_i \in \mathcal{B}_6 \setminus \{1\}$ using the entries of the matrix A, which are computed by the C++/SAGE program.

Let $b_i \in \mathcal{B}_6 \setminus \{1\}$. We first consider the case where $i > 12$. We write b_i as $z^k b_m$, for $k \in \{1,2,3\}$ and $b_m \in \mathcal{E}_6$. Since τ is a trace function, we have that $\tau(z^{4-k} b_m^{-1}) = \tau(z^{4-k} s^{p_1} t^{p_2})$, where $p_1 \in \{-1, 0\}$ and $p_2 \in \{-4,-3,-2,-1,0\}$. Using now the inverse Hecke relations, we can write $\tau(z^{4-k} s^{p_1} t^{p_2})$ as a $\mathbb{Z}[a, b^{\pm 1}, c, d, e^{\pm 1}]$-linear combination of elements of the form $\tau(z^{4-k} s^{q_1} t^{q_2})$, with $q_1 \in \{0,1\}$ and $q_2 \in \{0,1,2\}$. Since $k \in \{1,2,3\}$, we have that $z^{4-k} s^{q_1} t^{q_2} \in \mathcal{B}_6 \setminus \{1\}$ and, hence, $\tau(z^{4-k} b_m^{-1}) = 0$. We now consider the case where $i \leq 12$. We have:

$$\begin{aligned}
\tau(z^4 b_2^{-1}) &= \tau(z^3 ststs) = \tau(z^3 s^2 tst) = a\tau(z^3 stst) + b\tau(z^3 tst) = a\tau(b_{41} b_5) + b\tau(b_{45}) = 0. \\
\tau(z^4 b_3^{-1}) &= \tau(z^4 t^{-2}) = e^{-1}\tau(z^4 t) - ce^{-1}\tau(z^4) - de^{-1}\tau(z^4 t^{-1}) = e^{-1}\tau(b_{13} b_{38}) - ce^{-1}\tau(b_{25} b_{25}) - de^{-1}\tau(z^4 b_2^{-1}) = \\
&= e^{-1}(b^6 ce^4) - ce^{-1}(b^6 e^4) = 0. \\
\tau(z^4 b_4^{-1}) &= \tau(z^3 tstst) = \tau(b_{45} b_5) = 0. \\
\tau(z^4 b_5^{-1}) &= \tau(z^3 stst) = \tau(z^3 tsts) = \tau(b_{45} b_4) = 0. \\
\tau(z^4 b_6^{-1}) &= \tau(z^3 t^{-1} stst) = \tau(z^3 sts) = \tau(b_{40} b_7) = 0. \\
\tau(z^4 b_7^{-1}) &= \tau(z^4 s^{-1} t^{-1}) = \tau(z^4 t^{-1} s^{-1}) = \tau(z^4 b_5^{-1}) = 0. \\
\tau(z^4 b_8^{-1}) &= \tau(z^4 s^{-1} t^{-2}) = \tau(z^4 t^{-2} s^{-1}) = \tau(z^4 b_6^{-1}) = 0. \\
\tau(z^4 b_9^{-1}) &= \tau(z^4 t^{-1} s^{-1} t^{-1}) = \tau(z^4 t^{-2} s^{-1}) = \tau(z^4 b_6^{-1}) = 0. \\
\tau(z^4 b_{10}^{-1}) &= \tau(z^4 t^{-2} s^{-1} t^{-1}) = \tau(z^4 t^{-3} s^{-1}) = e^{-1}\tau(z^4 s^{-1}) - ce^{-1}\tau(z^4 t^{-1} s^{-1}) - de^{-1}\tau(z^4 t^{-2} s^{-1}) = \\
&= e^{-1}\tau(z^4 b_4^{-1}) - ce^{-1}\tau(z^4 b_5^{-1}) - de^{-1}\tau(z^4 b_6^{-1}) = 0. \\
\tau(z^4 b_{11}^{-1}) &= \tau(z^4 t^{-1} s^{-1} t^{-2}) = \tau(z^4 t^{-2} s^{-1} t^{-1}) = \tau(z^4 b_{10}^{-1}) = 0. \\
\tau(z^4 b_{12}^{-1}) &= \tau(z^4 t^{-2} s^{-1} t^{-2}) = \tau(z^4 t^{-4} s^{-1}) = e^{-1}\tau(z^4 t^{-1} s^{-1}) - ce^{-1}\tau(z^4 t^{-2} s^{-1}) - de^{-1}\tau(z^4 t^{-3} s^{-1}) = \\
&= e^{-1}\tau(z^4 b_5^{-1}) - ce^{-1}\tau(z^4 b_6^{-1}) - de^{-1}\tau(z^4 t^{-2} s^{-1} t^{-1}) = -de^{-1}\tau(z^4 b_{10}^{-1}) = 0.
\end{aligned}$$