The exceptional group G_7

1 Presentation

Let $R := \mathbb{Z}[a, b^{\pm 1}, c, d, e^{\pm 1}, f, g, h^{\pm 1}]$. The Hecke algebra associated with G_7 admits the following presentation:

$$H_{G_7} = \langle s, t, u \quad | \quad stu = tus = ust, \quad s^2 = as + b, \quad t^3 = ct^2 + dt + e, \quad u^3 = fu^2 + gu + h \rangle.$$

2 Relations

Braid relation : $stu = tus = ust$.

Positive Hecke relations : $s^2 = as + b, \quad t^3 = ct^2 + dt + e, \quad u^3 = fu^2 + gu + h$

Inverse Hecke relations : $s^{-1} = b^{-1}s - ab^{-1}, \quad t^{-1} = e^{-1}t^2 - ce^{-1}t - de^{-1}, \quad u^{-1} = h^{-1}u^2 - fh^{-1}u - gh^{-1}$

3 The basis $\mathcal{B}_7$

Let z be the central element stu. We recall that $|Z(G_7)| = 12$. We also set

$$\mathcal{E}_7 = \left\{1, u, u^2, t, t^2, ut, u^2t, ut^2, u^2t^2, tu^{-1}, tu^{-1}t, tu^{-1}t^2\right\}.$$

The basis $\mathcal{B}_7$ consisting of 144 elements is the following:

$b_1 = 1$	$b_{13} = z$	$b_{25} = z^2$	$b_{37} = z^3$
$b_2 = u$	$b_{14} = zu$	$b_{26} = z^2u$	$b_{38} = z^3u$
$b_3 = u^2$	$b_{15} = zu^2$	$b_{27} = z^2u^2$	$b_{39} = z^3u^2$
$b_4 = t$	$b_{16} = zt$	$b_{28} = z^2t$	$b_{40} = z^3t$
$b_5 = t^2$	$b_{17} = zt^2$	$b_{29} = z^2t^2$	$b_{41} = z^3t^2$
$b_6 = ut$	$b_{18} = zut$	$b_{30} = z^2ut$	$b_{42} = z^3ut$
$b_7 = u^2t$	$b_{19} = zu^2t$	$b_{31} = z^2u^2t$	$b_{43} = z^3u^2t$
$b_8 = ut^2$	$b_{20} = zut^2$	$b_{32} = z^2ut^2$	$b_{44} = z^3ut^2$
$b_9 = u^2t^2$	$b_{21} = zu^2t^2$	$b_{33} = z^2u^2t^2$	$b_{45} = z^3u^2t^2$
$b_{10} = \mathbf{tu^2}$	$b_{22} = ztu^{-1}$	$b_{34} = z^2tu^{-1}$	$b_{46} = z^3tu^{-1}$
$b_{11} = \mathbf{tu^2t}$	$b_{23} = ztu^{-1}t$	$b_{35} = z^2tu^{-1}t$	$b_{47} = z^3tu^{-1}t$
$b_{12} = \mathbf{tu^2t^2}$	$b_{24} = ztu^{-1}t^2$	$b_{36} = z^2tu^{-1}t^2$	$b_{48} = z^3tu^{-1}t^2$

$b_{49} = z^4$	$b_{61} = z^5$	$b_{73} = z^6$	$b_{85} = z^7$
$b_{50} = z^4u$	$b_{62} = z^5u$	$b_{74} = z^6u$	$b_{86} = z^7u$
$b_{51} = z^4u^2$	$b_{63} = z^5u^2$	$b_{75} = z^6u^2$	$b_{87} = z^7u^2$
$b_{52} = z^4t$	$b_{64} = z^5t$	$b_{76} = z^6t$	$b_{88} = z^7t$
$b_{53} = z^4t^2$	$b_{65} = z^5t^2$	$b_{77} = z^6t^2$	$b_{89} = z^7t^2$
$b_{54} = z^4ut$	$b_{66} = z^5ut$	$b_{78} = z^6ut$	$b_{90} = z^7ut$
$b_{55} = z^4u^2t$	$b_{67} = z^5u^2t$	$b_{79} = z^6u^2t$	$b_{91} = z^7u^2t$
$b_{56} = z^4ut^2$	$b_{68} = z^5ut^2$	$b_{80} = z^6ut^2$	$b_{92} = z^7ut^2$
$b_{57} = z^4u^2t^2$	$b_{69} = z^5u^2t^2$	$b_{81} = z^6u^2t^2$	$b_{93} = z^7u^2t^2$
$b_{58} = z^4tu^{-1}$	$b_{70} = z^5tu^{-1}$	$b_{82} = z^6tu^{-1}$	$b_{94} = z^7tu^{-1}$
$b_{59} = z^4tu^{-1}t$	$b_{71} = z^5tu^{-1}t$	$b_{83} = z^6tu^{-1}t$	$b_{95} = z^7tu^{-1}t$
$b_{60} = z^4tu^{-1}t^2$	$b_{72} = z^5tu^{-1}t^2$	$b_{84} = z^6tu^{-1}t^2$	$b_{96} = z^7tu^{-1}t^2$

$b_{97} = z^8$	$b_{109} = z^9$	$b_{121} = z^{10}$	$b_{133} = z^{11}$
$b_{98} = z^8u$	$b_{110} = z^9u$	$b_{122} = z^{10}u$	$b_{134} = z^{11}u$
$b_{99} = z^8u^2$	$b_{111} = z^9u^2$	$b_{123} = z^{10}u^2$	$b_{135} = z^{11}u^2$
$b_{100} = z^8t$	$b_{112} = z^9t$	$b_{124} = z^{10}t$	$b_{136} = z^{11}t$
$b_{101} = z^8t^2$	$b_{113} = z^9t^2$	$b_{125} = z^{10}t^2$	$b_{137} = z^{11}t^2$
$b_{102} = z^8ut$	$b_{114} = z^9ut$	$b_{126} = z^{10}ut$	$b_{138} = z^{11}ut$
$b_{103} = z^8u^2t$	$b_{115} = z^9u^2t$	$b_{127} = z^{10}u^2t$	$b_{139} = z^{11}u^2t$
$b_{104} = z^8ut^2$	$b_{116} = z^9ut^2$	$b_{128} = z^{10}ut^2$	$b_{140} = z^{11}ut^2$
$b_{105} = z^8u^2t^2$	$b_{117} = z^9u^2t^2$	$b_{129} = z^{10}u^2t^2$	$b_{141} = z^{11}u^2t^2$
$b_{106} = z^8tu^{-1}$	$b_{118} = z^9tu^{-1}$	$b_{130} = z^{10}tu^{-1}$	$b_{142} = z^{11}tu^{-1}$
$b_{107} = z^8tu^{-1}t$	$b_{119} = z^9tu^{-1}t$	$b_{131} = z^{10}tu^{-1}t$	$b_{143} = z^{11}tu^{-1}t$
$b_{108} = z^8tu^{-1}t^2$	$b_{120} = z^9tu^{-1}t^2$	$b_{132} = z^{10}tu^{-1}t^2$	$b_{144} = z^{11}tu^{-1}t^2$

3.1 Special cases

Case 1 : $u^{m_1}t^{n_1}z^ku^{m_2}t^{n_2}$, $k \in \{1, \ldots, 11\}$, $m_1, n_1 \in \{0, 1\}$ and $m_2, n_2 \in \{0, 1, 2\}$.

$$u^{m_1}t^{n_1}z^ku^{m_2}t^{n_2} = z^ku^{m_1}t^{n_1}u^{m_2}t^{n_2}.$$

Case 2 : $u^{m_1}t^{n_1}z^ktu^{-1}t^{n_2}$, $k \in \{1, \ldots, 11\}$, $m_1, n_1 \in \{0, 1\}$ and $n_2 \in \{0, 1, 2\}$.

$$u^{m_1}t^{n_1}z^ktu^{-1}t^{n_2} = z^ku^{m_1}t^{n_1+1}u^{-1}t^{n_2}.$$

Case 3 : $z^ktu^2t^n$, $k \in \{0, \ldots, 11\}$, $n \in \{0, 1, 2\}$.

$$z^ktu^2t^n = fz^ktut^n + gz^kt^{n+1} + hz^ktu^{-1}t^n.$$

Case 4 : $z^kt^mst^n$, $k \in \{0, \ldots, 10\}$, $m \in \{0, 1\}$, $n \in \{0, 1, 2\}$.

$$z^kt^mst^n = z^{k+1}t^mu^{-1}t^{n-1}.$$

Case 5 : $z^kt^mut^n$, $k \in \{0, \ldots, 9\}$, $m \in \{1, 2\}$, $n \in \{0, 1, 2\}$.

$$z^k t^m u t^n = z^{k+1} t^{m-1} s^{-1} t^n.$$

Case 6 : $z^k s u^m t^n$, $k \in \{1, \ldots, 10\}$, $m \in \{1,2\}$, $n \in \{0,1,2\}$.

$$z^k s u^m t^n = a z^k u^m t^n + b z^{k-1} t u^{m+1} t^n.$$

Case 7 : $z^k u^m t u^{-1} t^n$, $k \in \{0, \ldots, 7\}$, $m \in \{1,2\}$, $n \in \{0,1,2\}$.

$$\begin{aligned} z^k u^m t u^{-1} t^n &= -h^{-1} g z^k u^m t^{n+1} + h^{-1} b^{-1} z^{k+2} u^{m-1} t^{-1} u t^n + (-h^{-1} b^{-1} a) z^{k+1} u^{m+1} t^n + \\ &\quad + (-h^{-1} f b^{-1}) z^{k+2} u^{m-1} t^{n-1} + h^{-1} f b^{-1} a z^{k+1} u^m t^n. \end{aligned}$$

Case 8 : $z^k u t u^{-1} t^n$, $k \in \{8, \ldots, 11\}$, $n \in \{0,1,2\}$.

$$\begin{aligned} z^k u t u^{-1} t^n &= c z^k t^n + d a z^{k-1} u t^n + d b z^{k-2} u^2 t^{n+1} + f z^k t u^{-1} t^n + (-fc) z^k u^{-1} t^n + (-f d a) z^{k-1} t^n + \\ &\quad + (-f d b) z^{k-2} u t^{n+1} + e g a z^{k-2} s t^n + e g b z^{k-3} s u t^{n+1} + a e h z^{k-2} s u^{-1} t^n + a e h z^{k-1} u^{-2} t^{n-1} + \\ &\quad + (-a^2 e h) z^{k-2} u^{-1} t^n + g b^2 e h z^{k-4} u^{-1} t^{n+2} + f b e h z^{k-2} u^{-2} t^n + (-f b e h a) z^{k-3} u^{-1} t^{n+1} + \\ &\quad + b^2 e h^2 c z^{k-4} u^{-2} t^{n+1} + b^2 e h^2 d z^{k-5} s u^{-1} t^{n+1} + b^2 e^2 h^2 a z^{k-6} s t^{n+1} + b^3 e^2 h^2 z^{k-7} s u t^{n+2}. \end{aligned}$$

Case 9 : $z^k t^2 u^{-1} t^n$, $k \in \{2, \ldots, 11\}$, $n \in \{0,1,2\}$.

$$z^k t^2 u^{-1} t^n = c z^k t u^{-1} t^n + d z^k u^{-1} t^n + e a z^{k-1} t^n + e b z^{k-2} u t^{n+1}.$$

Case 10 : $z^k s t u^m t^n$, $k \in \{0, \ldots, 10\}$, $m, n \in \{0,1,2\}$.

$$z^k s t u^m t^n = z^{k+1} u^{m-1} t^n.$$

Case 11 : $z^k s t^2 u t^n$, $k \in \{0, \ldots, 8\}$, $n \in \{0,1,2\}$.

$$z^k s t^2 u t^n = b^{-1} z^{k+3} u^{-2} t^{n-1} - b^{-1} a z^{k+2} u^{-1} t^n.$$

Case 12 : $z^k t^2 u^{-1} t^n$, $k \in \{0,1\}$, $n \in \{0,1,2\}$.

$$\begin{aligned} z^k t^2 u^{-1} t^n &= -g h^{-1} z^k t^{n+2} - f h^{-1} z^k t^2 u t^n + (-h^{-1} b^{-1} a) z^{k+1} t u t^n + (-h^{-2} b^{-1} g) z^{k+2} u t^n + \\ &\quad + (-h^{-2} b^{-1} f) z^{k+3} s^{-1} t^{-1} u t^n + (-h^{-2} b^{-1} e^{-1} d) z^{k+3} s^{-1} u^2 t^n + h^{-2} b^{-2} e^{-1} c a z^{k+4} s^{-1} u t^n + \\ &\quad + (-h^{-2} b^{-2} e^{-1} c) z^{k+5} s^{-1} t^{n-1} + (-h^{-2} b^{-2} e^{-1} a) z^{k+4} s^{-1} u t^{n+1} + (-h^{-2} b^{-3} e^{-1} a) z^{k+5} t^n + \\ &\quad + h^{-3} b^{-3} e^{-1} z^{k+7} s^{-1} t^{-1} u t^{n-1} + (-h^{-3} b^{-3} e^{-1} f) z^{k+7} s^{-1} t^{n-2} + (-h^{-3} b^{-2} e^{-1} g) z^{k+5} s^{-1} u t^n + \\ &\quad + (-h^{-3} b^{-3} e^{-1} g a) z^{k+6} s^{-1} t^{n-1}. \end{aligned}$$

Case 13 : $z^k t u t^n$, $k \in \{10, 11\}$, $n \in \{0,1,2\}$.

$$\begin{aligned} z^k t u t^n &= f z^k t^{n+1} + g z^k t u^{-1} t^n + c h z^k u^{-2} t^n + a d h z^{k-1} u^{-1} t^n + b d h z^{k-2} u t u^{-1} t^n + h e a^2 z^{k-2} t^n + \\ &\quad + h e a b z^{k-3} u t^{n+1} + c h e b z^{k-2} t^{n-1} + d h e b a z^{k-3} t^{-1} u t^n + d h e b^2 g z^{k-4} t^n + d h e b^2 f z^{k-4} t^{-1} u t^{n+1} + \\ &\quad + d h^2 e b^2 a z^{k-5} t^{n+1} + d h^2 e b^3 z^{k-6} u t^{n+2} + h e^2 b a z^{k-3} t^{-1} u t^{n-1} + h e^2 b^2 f z^{k-4} t^{-2} u t^{n+1} + \\ &\quad + h e^2 b^3 g z^{k-6} u^2 t^{n+1} + h^2 e^2 b^2 a z^{k-6} s u t^{n+1} + h e^2 b^2 g a z^{k-5} t^{-1} u t^{n+1} + h^2 e^2 b^3 a z^{k-5} t^{-1} s^{-2} t^{n+1} + \\ &\quad + h^2 e^2 b^4 g z^{k-8} u t^{n+3} + h^2 e^2 b^3 f z^{k-6} u t u^{-1} t^n + (-h^2 e^2 b^3 f a) z^{k-7} u t^{n+2} + h^3 e^2 b^4 c z^{k-8} u t u^{-1} t^{n+1} + \\ &\quad + h^3 e^2 b^4 d z^{k-8} t^{n+1} + h^3 e^3 b^4 a z^{k-9} u t^{n+1} + h^3 e^3 b^5 z^{k-10} u^2 t^{n+2}. \end{aligned}$$

Case 14 : $t^2 u^2 t^n$, $n \in \{0,1,2\}$

$$t^2 u^2 t^n = f t^2 u t^n + g t^{n+2} + h t^2 u^{-1} t^n.$$

Case 15 : $u t u^2 t^n$, $n \in \{0,1,2\}$

$$u t u^2 t^n = f z^2 t^{-1} s^{-2} t^n + g u t^{n+1} + h u t u^{-1} t^n.$$

4 The extra condition

In order to prove the extra condition, we will write $\tau(z^{12}b_i^{-1})$ for all $b_i \in \mathcal{B}_7 \setminus \{1\}$ as $\mathbb{Z}[a, b^{\pm 1}, c, d, e^{\pm 1}, f, g, h^{\pm 1}]$-linear combinations of elements of the form $\tau(b_j b_l)$ with $b_j, b_l \in \mathcal{B}_7$ and elements of the form $\tau(z^{12}b_r^{-1})$ with $r \neq i$ that have already been calculated. Therefore, we can show that $\tau(z^{12}b_i^{-1}) = 0$ for all $b_i \in \mathcal{B}_7 \setminus \{1\}$ using the entries of the matrix A, which are computed by the C++/SAGE program, along with some inductive arguments.

Let $b_i \in \mathcal{B}_7 \setminus \{1\}$. We first consider the case where $i > 12$. We write b_i as $z^k b_m$, for $k \in \{1, \ldots, 11\}$ and $b_m \in \mathcal{E}_7$. Since τ is a trace function, we have that $\tau(z^{12-k}b_m^{-1}) = \tau(z^{12-k}u^{p_1}t^{p_2})$, where $p_1 \in \{-2, -1, 0, 1\}$ and $p_2 \in \{-3, -2, -1, 0\}$. Using now the inverse Hecke relations, we can write $\tau(z^{12-k}u^{p_1}t^{p_2})$ as a $\mathbb{Z}[a, b^{\pm 1}, c, d, e^{\pm 1}, f, g, h^{\pm 1}]$ -linear combination of elements of the form $\tau(z^{12-k}u^{q_1}t^{q_2})$, with $q_1, q_2 \in \{0, 1, 2\}$. Since $k \in \{1, \ldots, 11\}$, we have that $z^{12-k}u^{q_1}t^{q_2} \in \mathcal{B}_7 \setminus \{1\}$ and, hence, $\tau(z^{12-k}b_m^{-1}) = 0$. We now consider the case where $i \leq 12$. We have:

$$
\begin{aligned}
\tau(z^{12}b_2^{-1}) &= \tau(z^{11}st) = a\tau(z^{11}t) + b\tau(z^{11}s^{-1}t) = a\tau(b_{136}) + b\tau(z^{10}tut) = b\tau(z^{10}ut^2) = b\tau(b_{128}) = 0.\\
\tau(z^{12}b_3^{-1}) &= h^{-1}\tau(z^{12}u) - fh^{-1}\tau(z^{12}) - gh^{-1}\tau(z^{12}u^{-1}) = h^{-1}\tau(b_{13}b_{134}) - fh^{-1}\tau(b_{73}b_{73}) - gh^{-1}\tau(z^{12}b_2^{-1}) =\\
&= h^{-1}(b^6e^4fh^4) - fh^{-1}(b^6e^4h^4) = 0.\\
\tau(z^{12}b_4^{-1}) &= \tau(z^{11}us) = a\tau(z^{11}u) + b\tau(z^{11}us^{-1}) = a\tau(b_{134}) + b\tau(z^{10}utu) = b\tau(z^{10}u^2t) = b\tau(b_{127}) = 0.\\
\tau(z^{12}b_5^{-1}) &= e^{-1}\tau(z^{12}t) - ce^{-1}\tau(z^{12}) - de^{-1}\tau(z^{12}t^{-1}) = e^{-1}\tau(b_{13}b_{136}) - ce^{-1}\tau(b_{73}b_{73}) - de^{-1}\tau(z^{12}b_4^{-1}) =\\
&= e^{-1}(b^6ce^4h^4) - ce^{-1}(b^6e^4h^4) = 0.\\
\tau(z^{12}b_6^{-1}) &= \tau(z^{11}s) = a\tau(z^{11}) + b\tau(z^{11}s^{-1}) = a\tau(b_{133}) + b\tau(z^{10}tu) = b\tau(z^{10}ut) = b\tau(b_{126}) = 0.\\
\tau(z^{12}b_7^{-1}) &= \tau(z^{12}u^{-2}t^{-1}) = \tau(z^{11}u^{-1}s) = \tau(z^{11}su^{-1}) = a\tau(z^{11}u^{-1}) + b\tau(z^{11}s^{-1}u^{-1}) =\\
&= a\tau(z^{12}b_{14}^{-1}) + b\tau(z^{10}t) = b\tau(b_{124}) = 0.\\
\tau(z^{12}b_8^{-1}) &= \tau(z^{12}u^{-1}t^{-2}) = \tau(z^{11}st^{-1}) = \tau(z^{11}t^{-1}s) = a\tau(z^{11}t^{-1}) + b\tau(z^{11}t^{-1}s^{-1}) =\\
&= a\tau(z^{12}b_{16}^{-1}) + b\tau(z^{10}u) = b\tau(b_{122}) = 0.\\
\tau(z^{12}b_9^{-1}) &= \tau(z^{11}u^{-1}st^{-1}) = a\tau(z^{11}u^{-1}t^{-1}) + b\tau(z^{11}u^{-1}s^{-1}t^{-1}) = a\tau(z^{11}t^{-1}u^{-1}) + b\tau(z^{10}u^{-1}tut^{-1}) =\\
&= a\tau(z^{12}b_{18}^{-1}) + b\tau(z^{10}t^{-1}u^{-1}tu) = b\tau(z^9t^{-1}st^2u) = ab\tau(z^9tu) + b^2\tau(z^9t^{-1}s^{-1}t^2u) =\\
&= ab\tau(z^9ut) + b^2\tau(z^8ut^2u) = ab\tau(b_{114}) + b^2\tau(z^8u^2t^2) = b^2\tau(b_{105}) = 0.\\
\tau(z^{12}b_{10}^{-1}) &= \tau(z^{12}u^{-2}t^{-1}) = \tau(z^{12}t^{-1}u^{-2}) = \tau(z^{12}b_7^{-1}) = 0.\\
\tau(z^{12}b_{11}^{-1}) &= \tau(z^{12}t^{-1}u^{-2}t^{-1}) = \tau(z^{12}t^{-2}u^{-2}) = \tau(b^{12}b_9^{-1}) = 0.\\
\tau(z^{12}b_{12}^{-1}) &= \tau(z^{12}t^{-2}u^{-2}t^{-1}) = \tau(z^{12}t^{-3}u^{-2}) = e^{-1}\tau(z^{12}u^{-2}) - ce^{-1}\tau(z^{12}t^{-1}u^{-2}) - de^{-1}\tau(z^{12}t^{-2}u^{-2}) =\\
&= e^{-1}\tau(z^{12}b_3^{-1}) - ce^{-1}\tau(z^{12}b_7^{-1}) - de^{-1}\tau(z^{12}b_9^{-1}) = 0.
\end{aligned}
$$