The exceptional group G_8

1 Presentation

Let $R := \mathbb{Z}[a, b, c, d^{\pm 1}]$. The Hecke algebra associated with G_8 admits the following presentation:

$$H_{G_8} = \langle s, t \mid sts = tst, \quad s^4 = as^3 + bs^2 + cs + d, \quad t^4 = at^3 + bt^2 + ct + d \rangle.$$

2 Relations

Braid relation : $sts = tst$.

Positive Hecke relations : $s^4 = as^3 + bs^2 + cs + d, \quad t^4 = at^3 + bt^2 + ct + d$.

Inverse Hecke relations : $s^{-1} = d^{-1}s^3 - ad^{-1}s^2 - bd^{-1}s - cd^{-1}, \quad t^{-1} = d^{-1}t^3 - ad^{-1}t^2 - bd^{-1}t - cd^{-1}$.

3 The basis $\mathcal{B}_8$

Let z be the central element $ststst$. We recall that $|Z(G_8)| = 4$. We also set

$$\mathcal{E}_8 = \left\{1, s, s^2, s^3, t, ts, ts^2, ts^3, st, sts, sts^2, sts^3, s^2t, s^2ts, s^2ts^2, s^2ts^3, s^3t, s^3ts, s^3ts^2, s^3ts^3, t^2, st^2, s^2t^2, s^3t^2\right\}.$$

The basis $\mathcal{B}_8$ consisting of 96 elements is the following:

$b_1 = 1$	$b_{25} = z$	$b_{49} = z^2$	$b_{73} = z^3$
$b_2 = s$	$b_{26} = zs$	$b_{50} = z^2s$	$b_{74} = z^3s$
$b_3 = s^2$	$b_{27} = zs^2$	$b_{51} = z^2s^2$	$b_{75} = z^3s^2$
$b_4 = s^3$	$b_{28} = zs^3$	$b_{52} = z^2s^3$	$b_{76} = z^3s^3$
$b_5 = t$	$b_{29} = zt$	$b_{53} = z^2t$	$b_{77} = z^3t$
$b_6 = ts$	$b_{30} = zts$	$b_{54} = z^2ts$	$b_{78} = z^3ts$
$b_7 = ts^2$	$b_{31} = zts^2$	$b_{55} = z^2ts^2$	$b_{79} = z^3ts^2$
$b_8 = ts^3$	$b_{32} = zts^3$	$b_{56} = z^2ts^3$	$b_{80} = z^3ts^3$
$b_9 = st$	$b_{33} = zst$	$b_{57} = z^2st$	$b_{81} = z^3st$
$b_{10} = sts$	$b_{34} = zsts$	$b_{58} = z^2sts$	$b_{82} = z^3sts$
$b_{11} = sts^2$	$b_{35} = zsts^2$	$b_{59} = z^2sts^2$	$b_{83} = z^3sts^2$
$b_{12} = sts^3$	$b_{36} = zsts^3$	$b_{60} = z^2sts^3$	$b_{84} = z^3sts^3$
$b_{13} = s^2t$	$b_{37} = zs^2t$	$b_{61} = z^2s^2t$	$b_{85} = z^3s^2t$
$b_{14} = s^2ts$	$b_{38} = zs^2ts$	$b_{62} = z^2s^2ts$	$b_{86} = z^3s^2ts$
$b_{15} = s^2ts^2$	$b_{39} = zs^2ts^2$	$b_{63} = z^2s^2ts^2$	$b_{87} = z^3s^2ts^2$
$b_{16} = s^2ts^3$	$b_{40} = zs^2ts^3$	$b_{64} = z^2s^2ts^3$	$b_{88} = z^3s^2ts^3$
$b_{17} = s^3t$	$b_{41} = zs^3t$	$b_{65} = z^2s^3t$	$b_{89} = z^3s^3t$
$b_{18} = s^3ts$	$b_{42} = zs^3ts$	$b_{66} = z^2s^3ts$	$b_{90} = z^3s^3ts$
$b_{19} = s^3ts^2$	$b_{43} = zs^3ts^2$	$b_{67} = z^2s^3ts^2$	$b_{91} = z^3s^3ts^2$
$b_{20} = s^3ts^3$	$b_{44} = zs^3ts^3$	$b_{68} = z^2s^3ts^3$	$b_{92} = z^3s^3ts^3$
$b_{21} = t^2$	$b_{45} = zt^2$	$b_{69} = z^2t^2$	$b_{93} = z^3t^2$
$b_{22} = st^2$	$b_{46} = zst^2$	$b_{70} = z^2st^2$	$b_{94} = z^3st^2$
$b_{23} = s^2t^2$	$b_{47} = zs^2t^2$	$b_{71} = z^2s^2t^2$	$b_{95} = z^3s^2t^2$
$b_{24} = s^3t^2$	$b_{48} = zs^3t^2$	$b_{72} = z^2s^3t^2$	$b_{96} = z^3s^3t^2$

4 Special cases

Case 1 : $z^\ell s^\alpha t^m s^\beta z^k s^\gamma t^n s^\delta$ with $\alpha,\beta,\gamma,\delta,\ell\in\{0,1,2,3\}$, $m,n\in\{0,1\}$, and $k\in\{1,2,3\}$.

$$z^\ell s^\alpha t^m s^\beta z^k s^\gamma t^n s^\delta = z^{\ell+k} s^\alpha t^m s^{\beta+\gamma} t^n s^\delta.$$

Case 2 : $z^\ell s^\alpha t^m s^\beta z^k s^\gamma t^2$ with $\alpha,\beta,\gamma,\ell\in\{0,1,2,3\}$, $m,n\in\{0,1\}$, and $k\in\{1,2,3\}$.

$$z^\ell s^\alpha t^m s^\beta z^k s^\gamma t^2 = z^{\ell+k} s^\alpha t^m s^{\beta+\gamma} t^2$$

Case 3 : $z^\ell s^\gamma t^2 \cdot z^k s^\alpha t^m s^\beta$ with $\alpha,\beta,\gamma,\ell\in\{0,1,2,3\}$, $m,n\in\{0,1\}$, and $k\in\{1,2,3\}$.

$$z^\ell s^\gamma t^2 \cdot z^k s^\alpha t^m s^\beta = z^{\ell+k} s^\gamma t^2 s^\alpha t^m s^\beta.$$

Case 4 : $z^\ell s^\alpha t^2 \cdot z^k s^\beta t^2$ with $\alpha,\beta,\ell\in\{0,1,2,3\}$ and $k\in\{1,2,3\}$.

$$z^\ell s^\alpha t^2 \cdot z^k s^\beta t^2 = z^{\ell+k} s^\alpha t^2 s^\beta t^2.$$

Case 5 : $z^k s^m t^3 s^n$ with $k\in\{1,2,3\}$, $m,n\in\{0,1,2,3\}$.

$$z^k s^m t^3 s^n = a z^k s^m t^2 s^n + b z^k s^m t s^n + c z^k s^{m+n} + d z^{k-1} s^{m+2} t s^{n+2}$$

Case 6 : $z^k s^m t^2 s$ with $k\in\{0,1,2\}$ and $m\in\{0,1,2,3\}$.

$$z^k s^m t^2 s = z^{k+1} s^{m-1} t^{-2}.$$

Case 7 : $z^k s^m t s^2 t s^n$ with $k\in\{0,1,2\}$ and $m\in\{0,1,2,3\}$.

$$z^k s^m t s^2 t s^n = z^{k+1} s^{m-2+n}.$$

Case 8 : $z^k s^m t s^3 t s^n$ with $k\in\{0,1,2\}$ and $m\in\{0,1,2,3\}$.

$$z^k s^m t s^3 t s^n = z^{k+1} s^{m-1} t s^{n-1}.$$

Case 9 : $z^k s^m t^2 s^2$ with $k\in\{0,1\}$ and $m\in\{0,1,2,3\}$.

$$z^k s^m t^2 s^2 = d^{-1} z^{k+1} s^{m-1} t^2 s + (-ad^{-1}) z^{k+1} s^{m-1} t s + (-bd^{-1}) z^{k+1} s^m + (-cd^{-1}) z^k s^{m+1} t s^3.$$

Case 10 : $s^m t^2 s^3$ with $m\in\{0,1,2,3\}$.

$$s^m t^2 s^3 = d^{-1} z s^{m-1} t^2 s^2 + (-ad^{-1}) z s^{m-1} t s^2 + (-bd^{-1}) z s^{m+1} + (-cd^{-1}) s^{m+1} t s^4.$$

Case 11 : $z^k s^m t s^2 t^2 s^n$ with $k\in\{0,1,2\}$ and $m,n\in\{0,1,2,3\}$.

$$z^k s^m t s^2 t^2 s^n = z^{k+1} s^{m-2} t s^n.$$

Case 12 : $z^k s^m t s^3 t^2 s^n$ with $k\in\{0,1\}$ and $m,n\in\{0,1,2,3\}$.

$$z^k s^m t s^3 t^2 s^n = z^{k+1} s^{m-2} t^{-1} s t^2 s^n.$$

Case 13 : $z^k s^m t^2 s^2 t s^n$ with $k\in\{0,1,2\}$ and $m,n\in\{0,1,2,3\}$.

$$z^k s^m t^2 s^2 t s^n = z^{k+1} s^m t s^{n-2}.$$

Case 14 : $z^k s^m t^2 s^3 t s^n$ with $k \in \{0,1\}$ and $m,n \in \{0,1,2,3\}$.

$$z^{k+1}s^m ts^{-1}ts^{n-1} = d^{-1}z^{k+1}s^m ts^3ts^{n-1} + (-ad^{-1})z^{k+1}s^m ts^2ts^{n-1} + (-bd^{-1})z^{k+1}s^{m+1}ts^n + +(-cd^{-1})z^k s^{m+1}ts^4ts^n.$$

Case 15 : $z^k s^m t^2 s^2 t^2 s^n$ with $k \in \{0,1\}$ and $m,n \in \{0,1,2,3\}$.

$$z^k s^m t^2 s^2 t^2 s^n = z^{k+1}s^m ts^{-2}ts^n.$$

Case 16 : $z^k s^m t^3 s t^3 s^n$ with $k \in \{0,1\}$ and $m,n \in \{0,1,2,3\}$.

$$z^k s^m t^3 s t^3 s^n = z^{k+1}s^m ts^{-1}ts^n.$$

Case 17 : $s^m t^2 s^2 t^3$ with $m \in \{0,1,2,3\}$.

$$s^m t^2 s^2 t^3 = zs^{m-1}t^{-2}st^3.$$

Case 18 : $s^m t^2 s^3 t s^3 t^2 s^n$ with $m,n \in \{0,1,2,3\}$.

$$s^m t^2 s^3 t s^3 t^2 s^n = z^3 s^{m-1}t^{-5}s^{n-1}.$$

Case 19 : $s^m t^3 s^n$ with $m,n \in \{0,1,2,3\}$.

$$s^m t^3 s^n = zs^{m-1}t^{-1}st^{-1}s^{n-1}.$$

Case 20 : $z^k s^m t s^3 t^2 s^n$ with $k \in \{2,3\}$ and $m,n \in \{0,1,2,3\}$.

$$z^k s^m ts^3t^2s^n = az^k s^m ts^2t^2s^n + bz^k s^{m+2}ts^{n+1} + cz^k s^m t^3 s^n + dz^{k-1}s^m t^2 s^4 ts^{n+1}.$$

Case 21 : $z^3 s^m t s^2 t s^n$ with $m,n \in \{0,1,2,3\}$.

$$z^3 s^m ts^2ts^n = az^3 s^{m+1}ts^{n+1} + bz^3 s^m t^2 s^n + cz^2 s^m t^3 st^3 s^n + dz^2 s^m t^2 s^2 t^2 s^n.$$

Case 22 : $z^k s^m t^2 s^2 t^2 s^n$ with $k \in \{2,3\}$ and $m,n \in \{0,1,2,3\}$.

$$z^k s^m t^2 s^2 t^2 s^n = az^k s^{m+1}ts^2ts^n + bz^k s^m t^4 s^n + cz^{k-1}s^m t^4 st^4 s^n + dz^{k-1}s^m t^3 s^2 t^3 s^n.$$

Case 23 : $z^3 s^m t s^3 t s^n$ with $m,n \in \{0,1,2,3\}$.

$$z^3 s^m ts^3ts^n = az^3 s^m ts^2ts^n + bz^3 s^{m+1}ts^{n+1} + cz^3 s^m t^2 s^n + dz^2 s^m t^3 st^3 s^n.$$

Case 24 : $z^3 s^m t s^2 t^2 s^n$ with $m,n \in \{0,1,2,3\}$.

$$z^3 s^m ts^2t^2s^n = az^3 s^m ts^2ts^n + bz^3 s^m ts^{n+2} + cz^3 s^{m-1}t^2 s^{n+1} + dz^2 s^m ts^3t^2 s^{n+1}.$$

Case 25 : $z^k s^m t^3 s t^3 s^n$ with $k \in \{2,3\}$ and $m,n \in \{0,1,2,3\}$.

$$z^k s^m t^3 st^3 s^n = z^k s^{m+1}ts^3t^2s^n.$$

Case 26 : $z^k s^m t^2 s^3$ with $k \in \{1,2,3\}$ and $m \in \{0,1,2,3\}$.

$$z^k s^m t^2 s^3 = az^k s^m ts^3 + bz^k s^{m+3} + cz^{k-1}s^{m+2}ts^5 + dz^{k-1}s^{m+1}t^2 s^4.$$

Case 27 : $z^k s^m t^2 s^2$ with $k \in \{2,3\}$ and $m, n \in \{0,1,2,3\}$.

$$z^k s^m t^2 s^2 \quad = \quad a z^k s^m t s^2 + b z^k s^{m+2} + c z^{k-1} s^{m+2} t s^4 + d z^{k-1} s^{m+1} t^2 s^3.$$

Case 28 : $z^3 s^m t^2 s$ with $m \in \ 0,1,2,3\}$.

$$z^3 s^m t^2 s \quad = \quad a z^3 s^m t s + b z^3 s^{m+1} + c z^2 s^{m+2} t s^3 + d z^2 s^{m+1} t^2 s^2.$$

Case 29 : $z^k s^m t^3 s^2 t^3 s^n$ with $k \in \{1,2\}$ and $m, n \in \{0,1,2,3\}$.

$$z^k s^m t^3 s^2 t^3 s^n \quad = \quad a z^k s^m t^3 s^2 t^2 s^n + b z^{k+1} s^m t^2 s^{n-2} + c z^k s^m t^3 s^{n+2} + d z^{k-1} s^m t^4 s t^4 s^{n+1}.$$

5 The extra condition

In order to prove the extra condition, we will write $\tau(z^4 b_i^{-1})$ for all $b_i \in \mathcal{B}_8 \setminus \{1\}$ as $\mathbb{Z}[a,b,c,d^{\pm 1}]$-linear combinations of elements of the form $\tau(b_j b_l)$ with $b_j, b_l \in \mathcal{B}_8$ and elements of the form $\tau(z^4 b_r^{-1})$ with $r \neq i$ that have already been calculated. Therefore, we can show that $\tau(z^4 b_i^{-1}) = 0$ for all $b_i \in \mathcal{B}_8 \setminus \{1\}$ using the entries of the matrix A, which are computed by the C++/SAGE program.

Let $b_i \in \mathcal{B}_8 \setminus \{1\}$. We first consider the case where $i > 24$. We write b_i as $z^k b_m$, for $k \in \{1,2,3\}$ and $b_m \in \mathcal{E}_8$. Since τ is a trace function, we have that $\tau(z^{4-k} b_m^{-1}) = \tau(z^{4-k} s^{p_1} t^{p_2})$, where $p_1 \in \{-6,-5,-4,-3,-2,-1,0\}$ and $p_2 \in \{-2,-1,0\}$. Using now the inverse Hecke relations, we can write $\tau(z^{4-k} s^{p_1} t^{p_2})$ as a $\mathbb{Z}[a,b,c,d^{\pm 1}]$-linear combination of elements of the form $\tau(z^{4-k} s^{q_1} t^{q_2})$, with $q_1, q_2 \in \{0,1,2,3\}$. Since $k \in \{1,2,3\}$, we have that $z^{4-k} s^{q_1} t^{q_2} \in \mathcal{B}_8 \setminus \{1\}$ and, hence, $\tau(z^{4-k} b_m^{-1}) = 0$. We now consider the case where $i \leq 24$. We have:

$$\begin{aligned}
\tau(z^4 b_2^{-1}) &= \tau(z^3 t^2 s t^2) = \tau(z^3 s t^4) = \tau(b_{94} b_{21}) = 0.\\
\tau(z^4 b_3^{-1}) &= \tau(z^3 t s^2 t) = \tau(z^3 s^2 t^2) = \tau(b_{95}) = 0.\\
\tau(z^4 b_4^{-1}) &= \tau(z^2 t s^3 t s^3 t) = \tau(z^2 t^2 s^3 t s^3) = \tau(b_{69} b_{20}) = 0.\\
\tau(z^4 b_5^{-1}) &= \tau(z^3 s^2 t s^2) = \tau(b_{87}) = 0.\\
\tau(z^4 b_6^{-1}) &= \tau(z^3 s^2 t s) = \tau(b_{86}) = 0.\\
\tau(z^4 b_7^{-1}) &= \tau(z^3 t s^2) = \tau(b_{79}) = 0.\\
\tau(z^4 b_8^{-1}) &= \tau(z^3 t s t^{-1} s) = \tau(z^3 s t s t^{-1}) = \tau(z^3 t s) = \tau(b_{78}) = 0.\\
\tau(z^4 b_9^{-1}) &= \tau(z^4 t^{-1} s^{-1}) = \tau(z^4 s^{-1} t^{-1}) = \tau(z^4 b_6^{-1}) = 0.\\
\tau(z^4 b_{10}^{-1}) &= \tau(z^3 s t s) = \tau(b_{82}) = 0.\\
\tau(z^4 b_{11}^{-1}) &= \tau(z^3 t s) = \tau(b_{78}) = 0.\\
\tau(z^4 b_{12}^{-1}) &= \tau(z^3 s^{-1} t s) = \tau(z^3 t) = \tau(b_{77}) = 0.\\
\tau(z^4 b_{13}^{-1}) &= \tau(z^4 t^{-1} s^{-2}) = \tau(z^4 s^{-2} t^{-1}) = \tau(z^4 b_7^{-1}) = 0.\\
\tau(z^4 b_{14}^{-1}) &= \tau(z^3 s t) = \tau(b_{81}) = 0.\\
\tau(z^4 b_{15}^{-1}) &= \tau(z^3 t) = \tau(b_{77}) = 0.\\
\tau(z^4 b_{16}^{-1}) &= \tau(z^2 s t s^2 t^2) = \tau(b_{59} b_{21}) = 0.\\
\tau(z^4 b_{17}^{-1}) &= \tau(z^4 t^{-1} s^{-3}) = \tau(z^4 s^{-3} t^{-1}) = \tau(z^4 b_8^{-1}) = 0.\\
\tau(z^4 b_{18}^{-1}) &= \tau(z^4 s^{-1} t^{-1} s^{-3}) = \tau(z^4 s^{-2} t^{-1} s^{-2}) = \tau(z^4 b_{15}^{-1}) = 0.\\
\tau(z^4 b_{19}^{-1}) &= \tau(z^4 s^{-2} t^{-1} s^{-3}) = \tau(z^4 s^{-3} t^{-1} s^{-2}) = \tau(z^4 b_{16}^{-1}) = 0.\\
\tau(z^4 b_{20}^{-1}) &= \tau(z^3 s^{-1} t s^{-1}) = \tau(z^3 s^{-2} t) = \tau(z^2 t s^2 t^2) = \tau(b_{55} b_{21}) = 0.\\
\tau(z^4 b_{21}^{-1}) &= \tau(z^3 s t^2 s) = \tau(z^3 s^2 t^2) = \tau(b_{95}) = 0.\\
\tau(z^4 b_{22}^{-1}) &= \tau(z^3 s t^2) = \tau(b_{94}) = 0.\\
\tau(z^4 b_{23}^{-1}) &= \tau(z^3 s t^2 s^{-1}) = \tau(z^3 t^2) = \tau(b_{93}) = 0.\\
\tau(z^4 b_{24}^{-1}) &= \tau(z^2 s t^3 s^2 t) = \tau(z^2 t s t^3 s^2) = \tau(s^3 t s^3) = \tau(b_{68}) = 0.
\end{aligned}$$