Affine Nil-Hecke algebras

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Proposition

The group S_n admits a presentation with generators

$$s_1, ..., s_{n-1}$$

and relations

$$s_i^2 = 1,$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},$$

$$s_i s_j = s_j s_i, \text{ for } |i-j| > 1$$

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Mazumoto's Lemma

Let a a reduced decomposition of w. Then all reduced decompositions of w can be obtained from a by applying $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $s_i s_j = s_j s_i$, for $|i-j| > 1$

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Let $R_w = \{(i,j) \mid i < j \text{ and } w(i) > w(j)\}$. Then, $\ell(w) = |R_w|$.

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- The element $w_0 := (1, n)(2, n-1)(3, n-2)...$ is the unique element of S_n with maximal length $\ell(w_0) = \frac{n(n-1)}{2}$.

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$$(T_i - u_1)(T_i - u_2) = 0,$$

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- Given $w, w' \in S_n$ with $\ell(ww') = \ell(w) + \ell(w')$ we have $T_w T_{w'} = T_{ww'}$.

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See blackboard!

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It is a graded algebra with deg $T_i = -2$ and deg $X_i = 2$.

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Let $P_n = \mathbb{Z}[X_1, ..., X_n]$ and we let S_n act on P_n by permutation of the $X_i's$.

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Lemma

We have

$$\partial_i^2 = 0, \partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1} \text{ and } \partial_i\partial_j = \partial_j\partial_i, \text{ for } |i-j| > 1.$$

Representation of ${}^0H_n^f$ and ${}^0H_n$

Proposition

We obtain a representation of the nil Hecke algebra ${}^0H_n^f$ on P_n with the assignment

$$T_i \cdot P = \partial_i(P).$$

Given $w \in S_n$, denote ∂_w the image of T_w in $End_{\mathbb{Z}}(P_n)$.

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Proposition

We obtain a representation ρ of the affine nil Hecke algebra ${}^0H_n$ on P_n with the assignment

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Remark : This representation is compatible with the gradings. If $h_k \in ({}^0H_n)_k$ then

$$\rho(h_k)((P_n)_i) \subset (P_n)_{i+k}.$$

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This representation ρ of ${}^0H_n$ on P_n is faithful and we have a decomposition as $\mathbb{Z}$-modules :

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Let $a := \sum_w P_w T_w \in {}^0H_n$ ($a \neq 0$), with $\{P_w\}_{w \in S_n}$ a family in P_n.
We will find an element h of ${}^0H_n$ s.t. $\rho(a)(h) \neq 0$.

1. Take w_1 of minimal length s.t. $P_{w_1} \neq 0$, then by nice properties of the longest element w_0 :

$$ a\ T_{w_1{}^{-1}w_0} = \left(\sum_w P_w T_w \right) T_{w_1{}^{-1}w_0} = P_{w_1} T_{w_0}. $$

2. Compute the action of this element on $h := X_2 X_3^2 \cdots X_n^{n-1}$, we get

$$ \rho(a)(\partial_{w_1{}^{-1}w_0}(X_2 X_3^2 \cdots X_n^{n-1})) = P_{w_1} \neq 0. $$

Description of ${}^0H_n$ as a matrix ring over ${P_n}^{S_n}$

Theorem (Bernstein, Zelevinsky, Lusztig)

The center of the affine nil Hecke algebra is the ring of symmetric polynomials in $X_1, \ldots, X_n$: $Z({}^0H_n) = {P_n}^{S_n}$.

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General fact : from the morphism $\rho : {}^0H_n \longrightarrow End_{\mathbb{Z}}(P_n)$ we get a morphism of $Z({}^0H_n)$-algebras:

$$ {}^0H_n \longrightarrow End_{Z({}^0H_n)}(P_n). $$

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Sketch of proof for the isomorphism :

1. P_n is a progenerator as a ${}^0H_n$-module (f.g. projective ${}^0H_n$-module and ${}^0H_n$ is direct summand of a multiple of P_n as a ${}^0H_n$-module),
2. ${}^0H_n \longrightarrow End_{{P_n}^{S_n}}(P_n)$ is a split injection of ${P_n}^{S_n}$-modules,
3. ${}^0H_n$ is a free ${P_n}^{S_n}$-module of rank $(n!)^2$.

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1. P_n is a progenerator as a ${}^0H_n$-module (f.g. projective ${}^0H_n$-module and ${}^0H_n$ is direct summand of a multiple of P_n as a ${}^0H_n$-module),
2. ${}^0H_n \longrightarrow End_{{P_n}^{S_n}}(P_n)$ is a split injection of ${P_n}^{S_n}$-modules,
3. ${}^0H_n$ is a free ${P_n}^{S_n}$-module of rank $(n!)^2$.

Description of ${}^0H_n$ as a matrix ring

Since P_n is a free ${P_n}^{S_n}$-module of rank $n!$, the algebra ${}^0H_n$ is isomorphic to a $(n! \times n!)$-matrix algebra over ${P_n}^{S_n}$.